The threshold-two contact process on a random r-regular graph has a first order phase transition

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Abstract

We consider the discrete time threshold-two contact process on a random r-regular graph on n vertices. In this process, a vertex with at least two occupied neighbors at time t will be occupied at time $t+1$ with probability p, and vacant otherwise. We use a suitable isoperimetric inequality to show that if $r \geq 4$ and p is close enough to 1, then starting from all vertices occupied, there is a positive density of occupied vertices up to time $\exp(c(p)n)$ for some constant $c(p) > 0$. In the other direction, another appropriate isoperimetric inequality allows us to show that there is a decreasing function $\epsilon_2(p)$ and a constant $C_0(p) := 2/\log(2/(1+p))$ so that if the number of occupied vertices in the initial configuration is $\leq \epsilon_2(p)n$, then with high probability all vertexs are vacant at time $C_0(p)\log n$. These two conclusions imply that the density of occupied vertexs in the quasi-stationary distribution (defined in the paper) is discontinuous at the critical probability $p_c \in (0,1)$.

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1 Introduction

Interacting particle systems are often formulated on the d-dimensional integer lattice $\mathbf{Z}^d$. See e.g. [Lig85] or [Lig99]. However, if one is considering the spread of influenza in a town, infections occur not only between individuals who live close to each other, but also over long distances due to social contacts at school or at work. Because of this, one should consider how these stochastic spatial processes change when the regular lattice is replaced by the random graphs that have been used to model social networks.

[DJ07] considers the contact process on a small world graph $\mathcal{S}$. In the contact process, each vertex is either occupied or vacant. Occupied vertices become vacant at rate 1, while vacant vertices become occupied at rate λ times the number of occupied neighbors. The small world random graph, which [DJ07] considers, is a modification of the d-dimensional torus $\mathcal{T}_L := (\mathbf{Z} \bmod L)^d$ in which each vertex has exactly one long-distance neighbor, where the long-distance neighbors are defined by a random pairing of the vertices of the torus.

The contact process on the small world (or on any finite graph) cannot have a non-trivial stationary distribution, because it is a finite state Markov chain with an absorbing state. However, on the small world and many other graphs, there is a "quasi-stationary distribution" which persists for a long time. To explain the concept in quotes, we recall the situation for the contact process on the d-dimensional torus $\mathcal{T}_L$. Let $\zeta_t^0 \subseteq \mathbf{Z}^d$ denote the contact process on $\mathbf{Z}^d$ starting with single occupied vertex at the origin and let

$$\lambda_c := \inf\{\lambda : P(\Omega_\infty) > 0\}, \text{ where } \Omega_\infty := \{\zeta_t^0 \neq \emptyset \text{ for all } t\}.$$

Let $\zeta_t^1 \subseteq \mathbf{Z}^d$ denote the contact process on $\mathbf{Z}^d$ starting with all vertices occupied. Monotonicity and self-duality imply that (see [Lig99]) if $\lambda > \lambda_c$ and $\zeta_\infty^1 := \lim_{t\to\infty} \zeta_t^1$, where the limit is in distribution, then ζ_∞^1 is a translation invariant stationary distribution with $P(x \in \zeta_\infty^1) = P(\Omega_\infty)$.

Returning to the torus $\mathcal{T}_L$ and letting $\zeta_t^{1,\mathcal{T}_L} \subseteq \mathcal{T}_L$ denote the contact process on it starting from all vertices occupied, if $\lambda < \lambda_c$, then there is a $k_1(\lambda) > 0$ so that $P(\zeta_{k_1(\lambda)\log n}^{1,\mathcal{T}_L} \neq \emptyset) \to 0$ as $n \to \infty$, where $n = L^d$ is the number of vertices in $\mathcal{T}_L$. If $\lambda > \lambda_c$, then with high probability $\zeta_t^{1,\mathcal{T}_L}$ persists to time $\exp(k_2(\lambda)n)$ for some $k_2(\lambda) > 0$. Furthermore, at times $1 \ll t \leq \exp(k_2(\lambda)n)$ the finite dimensional distributions of $\zeta_t^{1,\mathcal{T}_L}$ are close to those of ζ_∞^1 (see [Lig99]). Thus the quasi-stationary distribution for the contact process on the finite graph is like the stationary distribution for the contact process on the associated infinite graph.

Locally, the small world graph $\mathcal{S}$ looks like an infinite graph that is called the big world $\mathcal{B}$ in [DJ07]. In this graph, traversing a long range edge brings one to another copy of $\mathbf{Z}^d$. Sophisticates will recognize this as the free product $\mathbf{Z}^d * \{0,1\}$, where the second factor is $\mathbf{Z} \bmod 2$. Like the contact process on the homogeneous tree, the contact process on $\mathcal{B}$ has two phase transitions $\lambda_1 < \lambda_2$, which correspond to global and local survival respectively. That is, if $\zeta_t^{0,\mathcal{B}} \subseteq \mathcal{B}$ denotes the contact process on $\mathcal{B}$ starting with single occupied vertex at

the origin, then

$$\lambda_1 := \inf\{\lambda : P(\Omega^{\mathcal{B}}_\infty) > 0\} \text{ and}$$
$$\lambda_2 := \inf\left\{\lambda : \liminf_{t\to\infty} P\left(0 \in \zeta^{0,\mathcal{B}}_t\right) > 0\right\},$$

where as earlier $\Omega^{\mathcal{B}}_\infty = \{\zeta^{0,\mathcal{B}}_t \neq \emptyset \text{ for all } t\}$. Let $\zeta^{1,\mathcal{B}}_t$ denote the contact process on $\mathcal{B}$ starting with all vertices occupied. Monotonicity and duality imply that if $\lambda > \lambda_1$ and $\zeta^{1,\mathcal{B}}_\infty := \lim_{t\to\infty} \zeta^{1,\mathcal{B}}_t$, where the limit is in distribution, then $\zeta^{1,\mathcal{B}}_\infty$ is a translation invariant stationary distribution with $P(x \in \zeta^{1,\mathcal{B}}_\infty) = P(\Omega^{\mathcal{B}}_\infty)$.

In order to study the persistence of the contact process $\zeta^{1,\mathcal{S}}_t \subseteq \mathcal{S}$ on the small world $\mathcal{S}$, [DJ07] introduces births at a rate γ from each vertex, which go from an occupied vertex to a randomly chosen vertex. With this modification it is shown that if $\lambda > \lambda_1$, then there is a constant $k_3 = k_3(\lambda, \gamma) > 0$ so that for $n = L^d$, $\zeta^{1,\mathcal{S}}_t$ persists to time $\exp(k_3 n)$ with high probability.

In this paper, we study the behavior of the discrete time *threshold-two contact process* on a random r-regular graph on n vertices. We construct our random graph G_n on the vertex set $V_n := \{1, 2, \dots n\}$ by assigning r "half-edges" to each of the vertices, and then pairing the half-edges at random. If r is odd, then n must be even so that the number of half-edges, rn, is even to have a valid degree sequence. Let $\mathbb{P}$ denote the distribution of G_n. We condition on the event E_n that the graph is simple, i.e. it does not contain a self-loop at any vertex, or more than one edge between two vertices. It can be shown (see e.g. Corollary 9.7 on page 239 of [JŁR00]) that $\mathbb{P}(E_n)$ is bounded away from 0, and hence for large enough n,

$$\text{if } \tilde{\mathbb{P}} := \mathbb{P}(\cdot|E_n), \text{ then } \tilde{\mathbb{P}}(\cdot) \le c\mathbb{P}(\cdot) \text{ for some constant } c = c(r) > 0. \tag{1.1}$$

So the conditioning on the event E_n will not have much effect on the distribution of G_n. Since the resulting graph remains the same under any permutation of the half-edges corresponding to any vertex, the distribution of G_n under $\tilde{\mathbb{P}}$ is uniform over the collection of all undirected r-regular graphs on the vertex set V_n. We choose G_n according to the distribution $\tilde{\mathbb{P}}$ on simple graphs, and once chosen the graph remains fixed through time.

We write $x \sim y$ to mean that x is a neighbor of y, and let

$$\mathcal{N}_y := \{x \in V_n : x \sim y\} \tag{1.2}$$

be the set of neighbors of y. The distribution $P_{G_n,p}$ of the (discrete time) threshold-two contact process $\xi_t \subseteq V_n$ with parameter p conditioned on G_n can be described as follows:

$$P_{G_n,p}(x \in \xi_{t+1} \mid |\mathcal{N}_x \cap \xi_t| \ge 2) = p \text{ and}$$
$$P_{G_n,p}(x \in \xi_{t+1} \mid |\mathcal{N}_x \cap \xi_t| < 2) = 0,$$

where the decisions for different vertices at time $t+1$ are taken independently. If $\mathbf{P}_p$ denotes the distribution of the threshold-two contact process ξ_t on the random graph G_n having distribution $\tilde{\mathbb{P}}$, then

$$\mathbf{P}_p(\cdot) = \tilde{\mathbb{E}} P_{G_n,p}(\cdot),$$

where $\tilde{\mathbb{E}}$ is the expectation corresponding to the probability distribution $\tilde{\mathbb{P}}$.

Let $\xi_t^A \subseteq V_n$ denote the threshold-two contact process starting from $\xi_0^A = A$, and let ξ_t^1 denote the special case when $A = V_n$. In the long history of the contact process the first step was to study whether the critical value of the parameter lies in the interior of the parameter-space or not. Based on results for the threshold contact process on random directed graph in [CD], and basic contact process on the small world $\mathcal{S}$ in [DJ07], it is natural to expect the existence of a critical value $p_c \in (0,1)$ defining the boundary between rapid convergence within logarithmically small time to all-zero configuration for $p < p_c$, and exponentially prolonged persistence of changes for $p > p_c$. We define the boundary p_c between convergence to the all-zero configuration within time $C(p)\log n$, and exponentially prolonged persistence as

$$p_c := \inf\left\{p \in [0,1] : \lim_{n\to\infty} \mathbf{P}_p\left(\inf_{t \le \exp(k(p)n)} \frac{|\xi_t^1|}{n} > u(p)\right) = 1 \text{ for some } k(p), u(p) > 0\right\}. \quad (1.3)$$

In order to show that $p_c < 1$, it suffices to show that if p is sufficiently close to 1, then ξ_t^1 maintains a positive fraction of occupied vertices for time $\ge \exp(c_1 n)$ for some constant $c_1 > 0$.

Theorem 1. *If $r \ge 4$ and $\eta \in (0, 1/4)$, then there is an $\epsilon_1 = \epsilon_1(\eta) \in (0,1)$ such that for*

$$\frac{1-\epsilon_1}{1 - \left(\frac{3}{2r-4} + \eta\right)\epsilon_1} < p \le 1, \quad (1.4)$$

and for some positive constants C_1 and $c_1(\eta, p)$,

$$\mathbf{P}_p\left(\inf_{t \le \exp(c_1(\eta,p)n)} \frac{|\xi_t^1|}{n} < 1 - \epsilon_1\right) \le C_1 \exp(-c_1(\eta,p)n).$$

In words, if p is sufficiently close to 1 and r is larger than 3, then the fraction of occupied vertices in the threshold-two contact process starting from all-one configuration remains close to 1 for exponentially long time with probability $1 - o(1)$. Here and later $o(1)$ denotes a quantity that goes to 0 as n goes to ∞. So Theorem 1 confirms that $p_c < 1$ for $r \ge 4$. The argument does not work for $r = 3$, as the lower bound in (1.4) is higher than 1 if we put $r = 3$. We believe that similar result holds for $r = 3$, but the problem remains open. The key to the proof of Theorem 1 is an 'isoperimetric inequality' (see Proposition 2 below).

Next we study the behavior of ξ_t^A, when $|A|$ is small.

Theorem 2. *There is a decreasing continuous function $\epsilon_2 : (0,1) \mapsto (0,1)$ and a collection $\mathcal{G}$ of simple r-regular graphs on n vertices such that for any $p \in (0,1)$, $C_0(p) := 2/\log(2/(1+p))$, and any subset $A \subset V_n$ with $|A| \le \epsilon_2(p)n$,*

$$\begin{array}{lll} (i) & \sup_{G_n \in \mathcal{G}} P_{G_n,p}\left(\xi^A_{\lceil C_0(p)\log n\rceil} \neq \emptyset\right) & = o(1), \\ (ii) & \tilde{\mathbb{P}}(\mathcal{G}^c) & = o(1). \end{array}$$

Hence $\lim_{n\to\infty} \mathbf{P}_p\left(\xi^A_{\lceil C_0(p)\log n\rceil} \neq \emptyset\right) = 0$.

In words, for any value of $p \in (0,1)$, whenever the fraction of occupied vertices drops below a certain level depending on p, all vertices of G_n become vacant within logarithmically small time with probability $1-o(1)$. Thus the density of occupied vertices doesn't stay in the interval $(0, \epsilon_2(p))$ for long time. The key to the proof of Theorem 2 is another 'isoperimetric inequality' (see Proposition 1 below). As a consequence of Theorem 2, we have:

Corollary 1. *There is a $p_0 \in (0, 2/3)$ such that for $0 \le p < p_0$,*

$$\lim_{n\to\infty} \mathbf{P}_p \left(\xi^1_{\lceil (C_0(p)+1)\log n \rceil} \neq \emptyset \right) = 0, \text{ where } C_0(p) \text{ is as in Theorem 2.}$$

That is, if p is sufficiently close to 0, then starting from all-one configuration all vertices of G_n become vacant within logarithmically small time with probability $1-o(1)$. So Corollary 1 confirms that $p_c > 0$.

Theorem 1 shows that $p_c < 1$, and so for $p \in (p_c, 1)$ the fraction of occupied vertices in the graph G_n is bounded away from zero for a time longer than $\exp(n^{1/2})$. So we can now define a quasi-stationary measure ξ^1_∞, which is an analogue of the upper invariant measure, as follows. For any $A \subset V_n$, $\xi^1_\infty\{B : B \cap A \neq \emptyset\} := \mathbf{P}_p(\xi^1_{\lceil \exp(n^{1/2}) \rceil} \cap A \neq \emptyset)$. Let X_n be uniformly distributed on V_n, and let

$$\rho_n := \xi^1_\infty\{B : X_n \in B\} = \frac{1}{n}\left|\xi^1_{\lceil \exp(n^{1/2}) \rceil}\right|.$$

So ρ_n is the quasi-stationary density of occupied vertices in the threshold-two contact process on the random graph G_n. Note that ρ_n is an analogue of the density of occupied vertices in the upper invariant measure for the contact process with sexual reproduction on regular lattices, which is conjectured to have a continuous phase transition (see Conjecture 1 and heuristic argument following that in [DN94]). As we now explain, things are different in the threshold-two contact process on a random regular graph.

First observe that if $p > p_c$, then ρ_n is bounded away from zero with high probability, because if $\rho_n < \epsilon_2(p)$, where $\epsilon_2(\cdot)$ is as in Theorem 2, then $|\xi^1_{\lceil \exp(n^{1/2}) \rceil}| \le n\epsilon_2(p)$. In that case, for $\sigma = \lceil \exp(n^{1/2}) \rceil + \lceil C_0 \log n \rceil$, either $\xi^1_\sigma \neq \emptyset$, which has $\mathbf{P}_p$-probability $o(1)$ by Theorem 2, or $\xi^1_\sigma = \emptyset$, which has $\mathbf{P}_p$-probability $o(1)$ by the definition of p_c in (1.3) and the fact that $p > p_c$. Therefore, for $p > p_c$, $\rho_n \ge \epsilon_2(p)$ with $\mathbf{P}_p$-probability $1 - o(1)$.

Next observe that for any $p_1, p_2 \in [0,1]$ with $p_1 < p_2$, the random variables $Z_i \sim$ *Bernoulli*$(p_i), i = 1, 2$, can be coupled so that $Z_1 \le Z_2$. Using this coupling for all the Bernoulli random variables, which are used in deciding whether $x \in \xi_t$ for $x \in V_n, t = 1, 2, \ldots$, it is easy to see that

$$P_{G_n,p_1} \le P_{G_n,p_2}, \text{ i.e. for any increasing event } B, P_{G_n,p_1}(B) \le P_{G_n,p_2}(B).$$

The same inequality holds for the unconditional probability distributions $\mathbf{P}_{p_1}$ and $\mathbf{P}_{p_2}$. Since $\{\rho_n \ge \epsilon\} = \{|\xi^1_{\lceil \exp(n^{1/2}) \rceil}| \ge \epsilon n\}$ is an increasing event, it follows that for any $p > p' > p_c$

$$\mathbf{P}_p(\rho_n \ge \epsilon_2(p')) \ge \mathbf{P}_{p'}(\rho_n \ge \epsilon_2(p')) = 1 - o(1)$$

by the above discussion. Taking p' sufficiently close to p_c and noting that $\epsilon_2(\cdot)$ is a decreasing continuous function, we get the result of this paper that the threshold-two contact process on the random graph G_n has a discontinuous phase transition at the critical value p_c.

Theorem 3. *Let $\rho := \epsilon_2(p_c)$, where $\epsilon_2(\cdot)$ is as in Theorem 2 and p_c is as in* (1.3). *Then $\rho > 0$. For any $p > p_c$ and $\delta > 0$,*

$$\lim_{n\to\infty} \mathbf{P}_p(\rho_n > \rho - \delta) = 1.$$

The key to the proof of Theorem 2 is an "isoperimetric inequality". Given a subset $U \subset V_n$, let

$$U^{*2} := \{y \in V_n : y \sim x \text{ and } y \sim z \text{ for some } x, z \in U \text{ with } x \neq z\}. \tag{1.5}$$

The idea behind this definition is that if $U = \xi_t$ for some t, then U^{*2} is the set of vertices which have a chance of being occupied at time $t+1$. Note that U^{*2} can contain vertices of U.

Proposition 1. *Let $E(m, k)$ be the event that there is a subset $U \subset V_n$ with size $|U| = m$ so that $|U^{*2}| \geq k$. Then there is an increasing positive function $\epsilon_3(\cdot)$ so that for any $\eta > 0$ and $m \leq \epsilon_3(\eta)n$,*

$$\mathbb{P}\left[E(m, (1+\eta)m)\right] \leq C_3 \exp\left(-\frac{\eta^2}{8r} m \log(n/m)\right)$$

for some constant $C_3 = C_3(r)$.

In words, if U is a small set, then for any $\eta > 0$, $|U^{*2}| \leq (1+\eta)|U|$ with high probability. Now if $E_{G_n,p}$ is the expectation corresponding to the probability distribution $P_{G_n,p}$, then $E_{G_n,p}(|\xi_{t+1}| \; |\xi_t) = p|\xi_t^{*2}|$. Given $p \in (0,1)$, we can choose $\eta(p) > 0$ so that $p(1+\eta(p)) < (1+p)/2$. So using Proposition 1, if $|\xi_t|$ is small, $E_{G_n,p}(|\xi_{t+1}| \; |\xi_t) < |\xi_t|(1+p)/2$ with high probability. This observation together with large deviation results for the Binomial distribution implies that $|\xi_{t+1}| \leq |\xi_t|(1+p)/2$ with high probability if $|\xi_t|$ is small. Finally if the number of occupied vertices reduces by a fraction at each time, all vertices will be vacant by time $O(\log n)$ and so Theorem 2 follows.

The key to the proof of Theorem 1 is another 'isoperimetric inequality'. If $W = V_n \setminus \xi_t$ is the set of vacant vertices at time t, then $(W^c)^{*2}$ is the set of vertices which have a chance of being occupied at time $t+1$, and so $((W^c)^{*2})^c$ is the set of vertices which will surely be vacant at time $t+1$.

Proposition 2. *Let $F(m, k)$ be the event that there is a subset $W \subset V_n$ with $|W| = m$ so that $|((W^c)^{*2})^c| > k$. Given $\eta > 0$, there are positive constants $\epsilon_4(r, \eta)$ and $C_4(r)$ so that for $m \leq \epsilon_4 n$,*

$$\mathbb{P}\left[F\left(m, \left(\frac{3}{2(r-2)} + \eta\right) m\right)\right] \leq C_4 \exp(-(\eta/8) m \log(n/m)).$$

In words, if W is a small set, then for any $\eta > 0$, $|((W^c)^{*2})^c| \le (3/(2r-4)+\eta)|W|$ with high probability. As noted above, $E_{G_n,p}(|\xi_{t+1}|\ |\xi_t) = p|\xi_t^{*2}|$. For p as in (1.4), we can choose $\delta(p,\eta) > 0$ so that $(p-\delta)(1-(3/(2r-4)+\eta)\epsilon_1) > 1-\epsilon_1$. So using Proposition 2 with $W = V_n \setminus \xi_t$, if $|\xi_t|/n \ge 1-\epsilon_1$, then $E_{G_n,p}(|\xi_{t+1}|\ |\xi_t) \ge p(1-(3/(2r-4)+\eta)\epsilon_1)n > (1-\epsilon_1)np/(p-\delta)$ with high probability. This observation together with large deviation results for the Binomial distribution implies that $|\xi_{t+1}| \le (1-\epsilon_1)n$ with exponentially small probability if $|\xi_t|/n \ge 1-\epsilon_1$. Thus if τ is the first time the fraction of occupied vertices drops below $1-\epsilon_1$, then $\tau > \exp(c_1(\eta,p)n)$ with high probability for a suitable choice of $c_1(\eta,p)$, and so Theorem 1 follows.

The remainder of the paper is organized as follows. In section 2, we present sketches of the proofs of Proposition 1 and 2. In section 3 and 4, we use the propositions to study the behavior of ξ_t starting from a small occupied set and the fact that $p_c \in (0,1)$ respectively, while in section 6 and 7 we present the proofs of the propositions. Section 5 is about the first order phase transition at p_c. Finally in section 8 we prove several probability estimates, which are needed in the proof of the propositions.

2 Sketch of the proofs of the isoperimetric inequalities.

Recall the definition of U^{*2} from (1.5). We need some more definitions and notations. For any vertex $x \in V_n$ and subsets $U, W \subset V_n$ let ∂U be the boundary of the set U, U^{*1} be the set of vertices which have at least one neighbor in U, $e(U,W)$ be the number of edges between U and W. Also let U_0 be the set of vertices in U which have all their neighbors in U^c, and U_1 be the complement of U_0. So

$$\begin{aligned} &\partial U := \{y \in U^c : y \sim x \text{ for some } x \in U\}, \quad U^{*1} := \{y \in V_n : y \sim x \text{ for some } x \in U\}, \\ &e(U,W) := |\{(x,y) : x \in U \text{ and } y \in W\}|, \\ &U_0 := \{x \in U : y \sim x \text{ implies } y \in U^c\}, \qquad U_1 := U \cap U_0^c. \end{aligned} \tag{2.1}$$

2.1 Isoperimetric inequality in Proposition 1

From the definitions in (2.1) it is easy to see that if $|U| = m$, then

$$rm \ge \sum_{y \in U^{*1}} e(\{y\},U) \ge |U^{*1} \setminus U^{*2}| + 2|U^{*2}| = |U^{*1}| + |U^{*2}|.$$

So for any subset U of vertices of size m,

$$\text{if } |U^{*2}| \ge k, \text{ then } |U^{*1}| \le rm - k. \tag{2.2}$$

In view of (2.2), for proving Proposition 1 it suffices to estimate the probability

$$\mathbb{P}\left[H(m,(r-1-\eta)m)\right], \text{ where } H(m,k) = \cup_{\{U \subset V_n : |U|=m\}} \left\{|U^{*1}| \le k\right\} \tag{2.3}$$

is the event that there is a subset U of vertices of size m with $|U^{*1}| \leq k$.

Note that U^{*1} is a disjoint union of ∂U and U_1. Our first step in estimating (2.3), taken in Lemma 8.2, is to show the following.

I. *For $|U| = m$ and any $\eta > 0$, $e(U, U^c) \geq (r-2-\eta)|U|$ with probability at least* $1 - \exp\left(-(1+\eta/2)m\log(n/m) + \Delta_1 m\right)$ *for some constant Δ_1.*

Take $\alpha = (r-2-\eta)/r$ in Lemma 8.2 so that $(1-\alpha)r/2 = 1+\eta/2$. We cannot hope to do better than $r-2$. Consider a tree in which all vertices have degree r and let U be the set of vertices within distance k of a fixed vertex. If $s = r-1$, then $|U| = 1+r+rs+\cdots+rs^{k-1} \approx rs^k/(s-1)$ and $e(U, U^c) = rs^k$, so $e(U, U^c)/|U| \approx s-1 = r-2$.

In the next step, see Lemma 8.4, we show the following.

II. *Given $e(U, U^c) = u|U|$ for some constant u and $\eta > 0$, if $m = |U| \leq \epsilon_5(\eta)n$, then $|\partial U| \geq (u-\eta)|U|$ with probability $\geq 1 - \exp\left(-\eta m\log(n/m) + \Delta_2 m\right)$ for some constant Δ_2.*

Considering all possible values of $u \geq r-2-\eta$ and using *I* and *II*,

$$|\partial U| \geq (r-2-2\eta)|U| \quad \text{with probability } \geq 1 - 2\exp\left(-(1+\eta)m\log(n/m) + (\Delta_1+\Delta_2)m\right).$$

Using the fact (see Lemma 8.1) that

III. *the number of subsets of V_n of size m is at most* $\exp(m\log(n/m) + m)$*,*

the expected number of subsets U of size m with $|\partial U| < (r-2-2\eta)|U|$ is exponentially small if $m \leq \epsilon(\eta)n$ for some small fraction $\epsilon(\eta)$. Therefore,

$$\text{with high probability } |U^{*1}| \geq |\partial U| \geq (r-2-2\eta)|U|, \text{ whenever } |U| \leq \epsilon(\eta)n. \tag{2.4}$$

But this is not good enough, so we need to work to improve the first inequality above.

Recall the definitions of U_0 and U_1 from (2.1). There are two possibilities based on $|U_1|$. Given $\eta > 0$, if $|U_1| \leq (\eta/2r)|U|$, then $e(U, U^c) \geq r|U_0| \geq (r-\eta/2)|U|$. So using *II*,

$$\begin{aligned}&\text{if } |U| = m, \text{ then } |\partial U| < (r-1-\eta)|U| \text{ and } |U_1| \leq (\eta/2r)|U| \text{ with probability at most}\\ &\exp\left(-(1+\eta/2)m\log(n/m) + \Delta_2 m\right).\end{aligned}$$

Combining with *III* the expected number of subsets of size m with the above property is exponentially small, if $m \leq \epsilon(\eta)n$. Therefore,

$$\text{with high probability } |U^{*1}| \geq |\partial U| \geq (r-1-\eta)|U| \quad \text{whenever } |U_1| \leq (\eta/2r)|U|.$$

Next we look at the other possibility $|U_1| > (\eta/2r)|U|$. Using an argument similar to the one leading to (2.4),

$$\text{with high probability } e(U_1, U_1^c) \geq (r-2-\eta)|U_1| \text{ whenever } |U| \leq \epsilon(\eta)n \text{ and } |U_1| > (\eta/2r)|U|. \tag{2.5}$$

Using the equalities $e(U_0, U^c) = e(U_0, U_0^c) = r|U_0|$ and $e(U_1, U^c) = e(U_1, U_1^c)$, we have $e(U, U^c) = r|U_0| + e(U_1, U_1^c)$. Combining this with another equality $|U^{*1}| = |U_1| + |\partial U|$ and a

little algebra give that $\{|U^{*1}| \le (r-1-\eta)|U|\} = \{e(U,U^c) - |\partial U| \ge (1+\eta)|U_0| + e(U_1, U_1^c) - (r-2-\eta)|U_1|\}$. In view of (2.5), the probability of the last events is estimated to be small enough (see (6.14) for details), so that using *III* the expected number of subsets U of size m with the above property is exponentially small. Combining the last two arguments,

$$\text{with high probability } |U^{*1}| \ge (r-1-\eta)|U| \quad \text{whenever } |U_1| > (\eta/2r)|U|.$$

This completes the argument to estimate the probability in (2.3) and thereby proves Proposition 1.

2.2 Isoperimetric inequality in Proposition 2

Recall the definition of $\mathcal{N}_y$ from (1.2). We need some more notations for Proposition 2. For any subset W of V_n, let W^0 be the subset of vertices which are in W and have at most 1 neighbor in W^c, and W^1 be the subset of vertices which are in W^c and have at most 1 neighbor in W^c. So

$$\begin{aligned} W^0 &:= \{y \in W : |\mathcal{N}_y \cap W| \ge r-1\}, \quad \beta_0(W) := |W^0|/|W|, \\ W^1 &:= \{y \in W^c : |\mathcal{N}_y \cap W| \ge r-1\}, \quad \beta_1(W) := |W^1|/|W|. \end{aligned} \tag{2.6}$$

The idea behind these definitions is that if W^c is occupied at tine t in the threshold-two contact process, then the subset of V_n, which cannot be occupied at time $t+1$, is

$$\left((W^c)^{*2}\right)^c = W^0 \cup W^1, \text{ and } \left|\left((W^c)^{*2}\right)^c\right| = |W^0| + |W^1|.$$

By *I*, $e(W^0, (W^0)^c) \ge (r-2-(2r-4)\eta)|W^0|$ with high probability if $|W| \le \epsilon(\eta)n$. But $e(W^0, W^c) \le |W^0|$ by the definition of W^0. So if $e(W^0, (W^0)^c) \ge (r-2-(2r-4)\eta)|W^0|$, then

$$\begin{aligned} e(W^0, W \setminus W^0) &= e(W_0, (W^0)^c) - e(W_0, W^c) \\ &\ge (r-2-(2r-4)\eta)|W^0| - |W^0|. \end{aligned}$$

Using $e(W^0, W^c) \le |W^0|$ again with $W_0 \subset W$ and the last inequality, we have

$$\begin{aligned} e(W, W^c) &= e(W \setminus W^0, W^c) + e(W^0, W^c) \\ &\le r|W \setminus W^0| - e(W \setminus W^0, W_0) + |W^0| \\ &= [r - (2r-4)(1-\eta)\beta_0(W)]|W|. \end{aligned}$$

Each $x \in \partial W$ has $e(\{x\}, W) \ge 1$ while each $x \in W^1$ has $e(\{x\}, W) \ge r-1$. So using the previous result and the definition of $\beta_i(W)$,

$$\begin{aligned} |\partial W| &\le e(W, W^c) - (r-2)|W^1| \\ &\le [r - (2r-4)(1-\eta)\beta_0(W) - (r-2)\beta_1(W)]|W|. \end{aligned}$$

Now if $(2r-4)(1-\eta)\beta_0(W)+(r-2)\beta_1(W) > 2+\eta$, then the above implies that $|\partial W| \le (r-2-\eta)|W|$, which has a small probability as mentioned earlier. From another viewpoint,

$$(r-2)|W^1| \le e(W,W^c) - |\partial W|. \tag{2.7}$$

By *II*, if $|W| = m$, then

$$e(W,W^c) - |\partial W| \le (1+2\eta)|W| \text{ with probability } \ge 1-\exp\left(-(1+2\eta)m\log(n/m)+\Delta_2 m\right),$$

and combining with *III* the expected number of subsets W with $e(W,W^c)-|\partial W| > (1+2\eta)|W|$ is exponentially small if $|W| \le \epsilon(\eta)n$. Therefore,

$$\text{with high probability } e(W,W^c)-|\partial W| \le (1+2\eta)|W| \text{ whenever } |W| \le \epsilon(\eta)n.$$

From (2.7), if $e(W,W^c)-|\partial W| \le (1+2\eta)|W|$, then $\beta_1(W) \le (1+2\eta)/(r-2)$.

Combining the last two observations, and noting that the maximum value of $\beta_0+\beta_1$ under the constraints (i) $2(1-\eta)\beta_0+\beta_1 \le (2+\eta)/(r-2)$ and (ii) $\beta_1 \le (1+2\eta)/(r-2)$ is achieved when both constraints are equalities, we see that with high probability

$$\beta_0+\beta_1 \le \frac{1}{2(r-2)} + \frac{1+2\eta}{r-2} \le \frac{3}{2(r-2)} + \frac{2}{r-2}\eta \le \frac{3}{2(r-2)} + \eta$$

for $r \ge 4$, and Proposition 2 is established.

3 Behavior of ξ_t starting from a small occupied set

In this section, we will use Proposition 1 to prove Theorem 2.

Proof of Theorem 2. If $p \in (0,1)$, we can choose $\eta > 0$ so that $(p+\eta)(1+\eta)$ equals any value between p and 1. To fix idea, we want to choose $\eta > 0$ so that $(p+\eta)(1+\eta) = (1+p)/2$. The roots of the quadratic equation $\eta^2+(1+p)\eta+p = (1+p)/2$ are $\eta_\pm = (-(1+p)\pm\sqrt{3+p^2})/2$. Clearly $\eta_- < 0$. Since $p \in (0,1)$, $(1+p)^2 \le 3+p^2$, which implies $(1+p) < \sqrt{3+p^2}$ and so $\eta_+ > 0$. We choose

$$\eta = \eta(p) := \frac{\sqrt{3+p^2}-(1+p)}{2} > 0 \text{ so that } (p+\eta)(1+\eta) = \frac{1+p}{2} < 1. \tag{3.1}$$

Next we take $\epsilon_2(p) := \epsilon_3(\eta(p))$, where $\epsilon_3(\cdot)$ is as in Proposition 1 and $\eta(\cdot)$ is as in (3.1). Since $\epsilon_3(\cdot)$ and $\eta(\cdot)$ are continuous, so is $\epsilon_2(\cdot)$. Also note that $\epsilon_3(\cdot)$ is increasing by Proposition 1, and

$$\frac{\partial\eta}{\partial p} = \frac{p}{2\sqrt{3+p^2}} - \frac{1}{2} < 0, \text{ as } p < \sqrt{3+p^2},$$

which implies that $\eta(\cdot)$ is decreasing. Combining these two observations, $\epsilon_2(\cdot)$ is decreasing. Having chosen ϵ_2, let

$$\mathcal{G} := \cap_{m=1}^{\lfloor\epsilon_2(p)n\rfloor} E_m^c, \text{ where } E_m = E(m,(1+\eta)m) \text{ is the event defined in Proposition 1.} \tag{3.2}$$

The argument for (i) consists of two steps.

Step 1: In the first step we show that for suitable choices of $C_{01} > 0$ and $b \in (0,1)$, if $|A| \le \epsilon_2 n$, then the number of occupied vertices in the threshold-two contact process ξ_t^A reduces to n^b within time $C_{01} \log n$. The argument of this step goes through for any choice of $b \in (0,1)$. But for future benefits we will choose b with the following desirable property.

First note that using the inequality $(1+p) < \sqrt{3+p^2}$,

$$\frac{\sqrt{3+p^2}}{2} < \frac{3+p^2}{2(1+p)} = \frac{(1+p)^2 + 2(1-p)}{2(1+p)}, \text{ which implies } \eta < \frac{1-p}{1+p}.$$

By the last inequality,

$$1+\eta < \frac{2}{1+p}, \text{ so that } \frac{\log(1+\eta)}{\log(2/(1+p))} < 1 \text{ and } \frac{\log(2/(1+p)) - \log(1+\eta)}{\log(2/(1+p)) + \log(1+\eta)} \in (0,1). \quad (3.3)$$

The assertion in (3.3) suggests that we can choose

$$b = b(p) \in (0,1) \text{ small enough, so that } b + (b+1)\frac{\log(1+\eta)}{\log(2/(1+p))} < 1 \text{ and } b \le \eta^2/16r. \quad (3.4)$$

Having chosen b, let A be any subset of vertices with $|A| \le \epsilon_2 n$, and define

$$\begin{aligned}
\nu &:= \min\left\{t : \left|\xi_t^A\right| \le n^b\right\}, \\
J_t &:= \left\{\left|\xi_t^A\right| \le \left(\frac{1+p}{2}\right)\left|\xi_{t-1}^A\right|\right\}, N_t := \cap_{s=1}^t J_s \text{ for } t \ge 1, N_0 := \{|\xi_0^A| \le \epsilon_2 n\}, \\
L_t &:= \left\{\text{ at most } (p+\eta)(1+\eta)\left|\xi_t^A\right| \text{ many vertices of } \left(\xi_t^A\right)^{*2} \text{ are occupied at time } t+1\right\}.
\end{aligned}$$

Now if L_t occur, then by the choice of η,

$$\left|\xi_{t+1}^A\right| \le (p+\eta)(1+\eta)\left|\xi_t^A\right| = \left(\frac{1+p}{2}\right)\left|\xi_t^A\right|. \text{ So } J_{t+1} \supset L_t. \quad (3.5)$$

By the definition of $(\xi_t^A)^{*2}$, each vertex of $(\xi_t^A)^{*2}$ will be in ξ_{t+1}^A with probability p, and for $G_n \in \mathcal{G}$, $|(\xi_t^A)^{*2}| \le (1+\eta)|\xi_t^A|$ on the event N_t. So using the binomial large deviations, see Lemma 2.3.3 on page 40 in [Dur07], and the stochastic monotonicity property of the Binomial distribution,

$$\begin{aligned}
P_{G_n,p}(L_t^c \cap N_t | \xi_t^A) &\le P(Binomial((1+\eta)|\xi_t^A|, p) > (p+\eta)(1+\eta)|\xi_t^A|) \\
&\le \exp\left(-\Gamma((p+\eta)/p)p(1+\eta)\left|\xi_t^A\right|\right), \quad (3.6)
\end{aligned}$$

where $\Gamma(x) = x\log x - x + 1 > 0$ for $x \ne 1$. Since $|\xi_t^A| \ge n^b$ on $\{t < \nu\}$, we can replace $|\xi_t^A|$ in the right side of (3.6) by n^b to have

$$P_{G_n,p}(L_t^c \cap N_t \cap \{t < \nu\}) \le P_{G_n,p}\left(L_t^c \cap N_t \cap \left\{\left|\xi_t^A\right| \ge n^b\right\}\right) \le \exp(-\Gamma((p+\eta)/p)pn^b). \quad (3.7)$$

Combining (3.5) and (3.7) we get

$$\begin{aligned} P_{G_n,p}(J^c_{t+1} \cap N_t \cap \{t < \nu\}) &\le P_{G_n,p}(L^c_t \cap N_t \cap \{t < \nu\}) \\ &\le \exp(-\Gamma((p+\eta)/p)\, pn^b). \end{aligned} \tag{3.8}$$

We choose

$$C_{01}(p) := (1-b(p))/\log(2/(1+p)) \;\text{ to satisfy }\; \left(\frac{1+p}{2}\right)^{C_{01}\log n} n = n^b, \tag{3.9}$$

so that $N_{\lceil C_{01}\log n\rceil} \subset \{|\xi^A_{\lceil C_{01}\log n\rceil}| \le [(1+p)/2]^{C_{01}\log n}|A| < n^b\}$. Hence $\{\nu > \lceil C_{01}\log n\rceil\} \subset N^c_{\lceil C_{01}\log n\rceil}$. Therefore, recalling the definition of N_t and noting that N^c_t is the disjoint union $\cup_{s=1}^t (J^c_s \cap N_{s-1})$,

$$\begin{aligned} P_{G_n,p}(\nu > \lceil C_{01}\log n\rceil) &= P_{G_n,p}\left(\{\nu > \lceil C_{01}\log n\rceil\} \cap N^c_{\lceil C_{01}\log n\rceil}\right) \\ &\le P_{G_n,p}\left[\cup_{t=1}^{\lceil C_{01}\log n\rceil}(J^c_t \cap N_{t-1} \cap \{\nu > t-1\})\right] \\ &\le \sum_{t=1}^{\lceil C_{01}\log n\rceil} P_{G_n,p}(J^c_t \cap N_{t-1} \cap \{\nu > t-1\}). \end{aligned}$$

Using (3.8) we can bound the summands of the above sum, and have

$$P_{G_n,p}(\nu > \lceil C_{01}\log n\rceil) \le \lceil C_{01}\log n\rceil \exp(-\Gamma((p+\eta)/p)\, pn^b) \le \exp(-\Gamma((p+\eta)/p) pn^b/2) \tag{3.10}$$

for large enough n.

Step2: Our next goal is to show that starting from any subset B of size $|B| \le n^b$, the threshold-two contact process ξ^B_t dies out within time $C_{02}\log n$ for a suitable choice of $C_{02} > 0$. Note that we always have $|\xi^B_{t+1}| \le |(\xi^B_t)^{*2}|$. In addition, for $G_n \in \mathcal{G}$ we have $|(\xi^B_t)^{*2}| \le (1+\eta)|\xi^B_t|$ only when $|\xi^B_t| \le \epsilon_2(p)n$. Keeping this in mind, we recall the choice of b from (3.4) and choose

$$C_{02}(p) := (b+1)/\log(2/(1+p)) \text{ to satisfy } b + C_{02}\log(1+\eta) < 1, \tag{3.11}$$

so that for $G_n \in \mathcal{G}$ and $t \le C_{02}\log n$ and large enough n,

$$\left|\xi^B_t\right| \le (1+\eta)\left|\xi^B_{t-1}\right| \le \cdots \le (1+\eta)^t\left|\xi^B_0\right| \le (1+\eta)^t n^b \le n^{b+C_{02}\log(1+\eta)} < \epsilon_2(p)n.$$

Now if $\mathcal{F}_t = \sigma\{\xi^B_s : 0 \le s \le t\}$, then

$$\begin{gathered} E_{G_n,p}\left(\left|\xi^B_{t+1}\right| \,\middle|\, \mathcal{F}_t\right) = p\left|\left(\xi^B_t\right)^{*2}\right|, \text{ and so} \\ \text{for } t \le C_{02}\log n \text{ and } G_n \in \mathcal{G}, E_{G_n,p}\left(\left|\xi^B_{t+1}\right| \,\middle|\, \mathcal{F}_t\right) \le p(1+\eta)|\xi^B_t|. \end{gathered} \tag{3.12}$$

Iterating the above inequality,

$$E_{G_n,p}\left(\left|\xi^B_{\lceil C_{02}\log n\rceil}\right|\right) \le [p(1+\eta)]^{C_{02}\log n}|\xi^B_0| \text{ for } G_n \in \mathcal{G}.$$

Now by the choices of η in (3.3), $p(1+\eta) < (1+p)/2$, and by the choice of C_{02} in (3.11), $[(1+p)/2]^{C_{02}\log n} = n^{-(1+b)}$. So

$$[p(1+\eta)]^{C_{02}\log n}|\xi_0^B| \le \left(\frac{1+p}{2}\right)^{C_{02}\log n} n^b = 1/n.$$

Combining the last two inequalities,

$$E_{G_n,p}\left(\left|\xi^B_{\lceil C_{02}\log n\rceil}\right|\right) \le \frac{1}{n} \text{ for } G_n \in \mathcal{G}.$$

Finally using Markov inequality,

$$P_{G_n,p}\left(\left|\xi^B_{\lceil C_{02}\log n\rceil}\right| \ge 1\right) \le E_{G_n,p}\left(\left|\xi^B_{\lceil C_{02}\log n\rceil}\right|\right) \le \frac{1}{n} \text{ for } G_n \in \mathcal{G}.$$

Combining with (3.10), and using the Markov property of the threshold-two contact process under the probability distribution $P_{G_n,p}$, we get the result in (i) for $C_0(p) := C_{01}(p)+C_{02}(p)$, where C_{01} is as in (3.9) and C_{02} is as in (3.11).

To show (ii) we use Proposition 1 and the fact from (1.1) that $\tilde{\mathbb{P}}(\cdot) \le c\mathbb{P}(\cdot)$ to have

$$\tilde{\mathbb{P}}(\mathcal{G}^c) \le \sum_{m=1}^{\lfloor \epsilon_2(p)n\rfloor} \tilde{\mathbb{P}}(E_m) \le cC_3\left[\sum_{m=\lceil n^b\rceil}^{\lfloor \epsilon_2(p)n\rfloor} \exp\left(-\frac{\eta^2}{8r}m\log\frac{n}{m}\right) + \sum_{m=1}^{\lceil n^b\rceil -1} \exp\left(-\frac{\eta^2}{8r}m\log\frac{n}{m}\right)\right].$$

Noting that the function $\phi(\eta) = \eta\log(1/\eta)$ is increasing for $\eta \in (0,1/e)$ (see (8.2)) and recalling that $\epsilon_2(p) \le 1/e$ by its definition, $m\log(n/m) = n\phi(m/n)$ is an increasing function of m for $m \le \epsilon_2(p)n$. So we can bound the summands of the last display by the first terms of the respective sums to have

$$\tilde{\mathbb{P}}(\mathcal{G}^c) \le cC_3\left[(n-n^b)\exp\left(-\frac{\eta^2}{8r}n^b\log(n/n^b)\right) + n^b\exp(-(\eta^2/8r)\log n)\right] = o(1),$$

as $b \le \eta^2/16r$ by our choice in (3.4). $\square$

4 The critical value p_c

In this section, we show that the critical value p_c is in the interval $(0,1)$. The fact that $p_c > 0$ follows as a consequence of Theorem 2.

Proof of Corollary 1. If $\mathcal{H}_t := \sigma\{\xi^1_s : 0 \le s \le t\}$, then, as observed in (3.12), $E_{G_n,p}(|\xi^1_{t+1}|\,|\mathcal{H}_t) = p|(\xi^1_t)^{*2}| \le np$. So using Markov inequality,

$$\text{if } K_t := \{|\xi^1_t| \ge 3np/2\}, \text{ then } P_{G_n,p}(K_{t+1}|\mathcal{H}_t) \le \frac{2}{3}.$$

Using properties of the conditional expectation,

$$E_{G_n,p}\left(\mathbf{1}_{\cap_{s=1}^{t+1}K_s}\middle|\mathcal{H}_t\right) = \mathbf{1}_{\cap_{s=1}^{t}K_s} E_{G_n,p}(\mathbf{1}_{K_{t+1}}|\mathcal{H}_t) \le \frac{2}{3}\mathbf{1}_{\cap_{s=1}^{t}K_s},$$

so that $E_{G_n,p}\mathbf{1}_{\cap_{s=1}^{t+1}K_s} \le \frac{2}{3}E_{G_n,p}\mathbf{1}_{\cap_{s=1}^{t}K_s}$. Iterating the last inequality,

$$P_{G_n,p}(\cap_{s=1}^{\lfloor \log n\rfloor}K_s) \le (2/3)^{\lfloor \log n\rfloor} \le (3/2)n^{-\log(3/2)}. \tag{4.1}$$

Now since $\epsilon_2 : (0,1) \mapsto (0,1)$ is decreasing and continuous, by intermediate value theorem there is a unique $p_0 \in (0, 2/3)$ such that $\epsilon_2(p_0) = 3p_0/2$ and for $p \in [0, p_0)$, $\epsilon_2(p) > 3p/2$. So if $p \in [0, p_0)$, then (4.1) suggests that $|\xi_s^1|/n$ drops below $\epsilon_2(p)$ for some $s \le \log n$ with $P_{G_n,p}$-probability $\ge 1 - (3/2)n^{-\log(3/2)}$. Combining this with (i) of Theorem 2, noting that $\lfloor \log n\rfloor + \lceil C_0(p)\log n\rceil \le \lceil (C_0(p) + 1)\log n\rceil$, and using Markov property of $P_{G_n,p}$, we have

$$\sup_{G_n\in\mathcal{G}} P_{G_n,p}\left(\xi^1_{\lceil (C_0(p)+1)\log n\rceil} \ne \emptyset\right) = o(1) \text{ for } p \in [0, p_0) \text{ and } G_n \in \mathcal{G}.$$

This together with (ii) of Theorem 2 proves the desired result. □

Now we show that $p_c < 1$ using Proposition 2.

Proof of Theorem 1. Given $\eta \in (0, 1/4)$ let $\epsilon_4(\eta)$ be the constant in Proposition 2 and take $\epsilon_1 := \epsilon_4$. Since $r \ge 4$ and $\eta < 1/4$, $3/(2r - 4) \le 3/4 < 1 - \eta$ so that the fraction in (1.4) is < 1. For p between this fraction and 1, we can choose $\delta = \delta(\eta, p) > 0$ such that

$$(p - \delta)\left(1 - \left(\frac{3}{2r-4} + \eta\right)\epsilon_1\right) > 1 - \epsilon_1. \tag{4.2}$$

For $t = 0, 1, \ldots$ if $|\xi_t^1| \le \lfloor (1 - \epsilon_1/2)n\rfloor$, then let $U_t = \xi_t^1$, and if $|\xi_t^1| > \lfloor (1 - \epsilon_1/2)n\rfloor$, we have too many vertices to use Proposition 2, so we let U_t be the subset of ξ_t^1 consisting of $\lfloor (1 - \epsilon_1/2)n\rfloor$ many vertices with smallest indices. Thus $|U_t^c| \ge \epsilon_1 n/2$ for any $t \ge 0$. We begin with some notations. For $t \ge 0$ let

$$\begin{aligned}
I_t &:= \left\{|\xi_t^1| \ge (1 - \epsilon_1)n\right\}, \quad O_t := \cap_{s=0}^t I_s,\\
S_t &:= \left\{\left|U_t^{*2}\right| \ge n - \left(\frac{3}{2r-4} + \eta\right)|U_t^c|\right\},\\
T_t &:= \left\{\text{ at least } (p - \delta)\left|U_t^{*2}\right| \text{ many vertices of } U_t^{*2} \text{ are occupied at time } t + 1\right\}.
\end{aligned}$$

On the event $S_t \cap T_t$, $|\xi_{t+1}^1| \ge (p - \delta)|U_t^{*2}| \ge (p - \delta)[n - (3/(2r - 4) + \eta)|U_t^c|]$, and on the event O_t, $|\xi_t^1| \ge (1 - \epsilon_1)n$ so that $|U_t| = \min\{|\xi_t^1|, \lfloor (1 - \epsilon_1/2)n\rfloor\} \ge (1 - \epsilon_1)n$ and hence $|U_t^c| \le \epsilon_1 n$. Therefore, using (4.2) it is easy to see that on the event $S_t \cap T_t \cap O_t$,

$$\left|\xi_{t+1}^1\right| \ge (p - \delta)\left(1 - \left(\frac{3}{2r-4} + \eta\right)\epsilon_1\right)n > (1 - \epsilon_1)n.$$

So $I_{t+1} \cap O_t \supset S_t \cap T_t \cap O_t$ for any $t \geq 0$. Next we see that if we take $F_t := F(|U_t^c|, (3/(2r-4)+\eta)|U_t^c|)$, where $F(\cdot,\cdot)$ is defined in Proposition 2, then $P_{G_n,p}(S_t|U_t) \geq \mathbf{1}_{F_t^c}$, since $|(U_t^{*2})^c| \leq (3/(2r-4)+\eta)|U_t^c|$ on the event S_t. Taking expectation with respect to the distribution of G_n, $\mathbf{P}_p(S_t|U_t) \geq \tilde{\mathbb{P}}(F_t^c)$. As noted above, $|U_t^c| \leq \epsilon_1 n$ on the event O_t. So, recalling from (1.1) that $\tilde{\mathbb{P}}(\cdot) \leq c\mathbb{P}(\cdot)$, we can apply Proposition 2 with $m = |U_t^c|$ to have

$$\mathbf{P}_p(S_t^c \cap O_t|U_t) \leq \mathbf{P}_p(S_t^c \cap \{|U_t^c| \leq \epsilon_1 n\}|U_t) \leq cC_4 \exp(-(\eta/8)|U_t^c| \log(n/|U_t^c|)).$$

Since $\epsilon_1 = \epsilon_4 \leq 1/e$ by (7.10), combining the facts that the function $\phi(\eta) = \eta \log(1/\eta)$ is increasing on $(0, 1/e)$ (see (8.2)) and $|U_t^c|$ is always $\geq \epsilon_1 n/2$ by its definition, we have $\phi(|U_t^c|/n) > \phi(\epsilon_1/2)$ or equivalently $|U_t^c| \log(n/|U_t^c|) \geq (\epsilon_1/2)n \log(2/\epsilon_1)$ on the event O_t. Keeping this in mind, we can increase the upper bound in the last display to have

$$\mathbf{P}_p(S_t^c \cap O_t) \leq \mathbf{P}_p\left(S_t^c \cap \{\epsilon_1 n/2 \leq |U_t^c| \leq \epsilon_1 n\}\right) \leq cC_4 \exp\left(-\frac{\eta}{8}\frac{\epsilon_1}{2}\log(2/\epsilon_1)n\right). \tag{4.3}$$

On the other hand, using the binomial large deviation, see Lemma 2.3.3 on page 40 in [Dur07],

$$P_{G_n,p}(T_t \mid U_t^{*2}) \geq 1 - \exp\left(-\Gamma((p-\delta)/p)p\left|U_t^{*2}\right|\right), \tag{4.4}$$

where $\Gamma(x) = x\log x - x + 1 > 0$ for $x \neq 1$. As noted earlier in the proof, on the event O_t, $|\xi_t^1| \geq (1-\epsilon_1)n$ so that $|U_t| = \min\{|\xi_t^1|, \lfloor(1-\epsilon_1/2)n\rfloor\} \geq (1-\epsilon_1)n$. Therefore, on the event $S_t \cap O_t$, $|U_t^{*2}| \geq [1-(3/(2r-4)+\eta)\epsilon_1]n$. Keeping this in mind, we can replace $|U_t^{*2}|$ in the right hand side of (4.4) by $[1-(3/(2r-4)+\eta)\epsilon_1]n$ to have

$$\begin{aligned} P_{G_n,p}(T_t^c \cap S_t \cap O_t) &\leq P_{G_n,p}(T_t^c \cap \{|U_t^{*2}| \geq [1-(3/(2r-4)+\eta)\epsilon_1]n\}) \\ &\leq \exp\left(-\Gamma((p-\delta)/p)p\left\{1 - \left(\frac{3}{2r-4}+\eta\right)\epsilon_1\right\}n\right). \end{aligned} \tag{4.5}$$

The same bound also works for the unconditional probability distribution $\mathbf{P}_p$. Combining these two bounds of (4.3) and (4.5), and recalling that $I_{t+1} \cap O_t \supset S_t \cap T_t \cap O_t$,

$$\mathbf{P}_p(I_{t+1}^c \cap O_t) \leq \mathbf{P}_p((S_t \cap T_t)^c \cap O_t) \leq \mathbf{P}_p(S_t^c \cap O_t) + \mathbf{P}_p(T_t^c \cap S_t \cap O_t) \leq C_1 \exp(-2c_1(\eta,p)n),$$

where $C_1 = 2\max\{1, cC_4\}$ and

$$c_1(\eta,p) = \frac{1}{2}\min\left\{\frac{\eta\epsilon_1}{16}\log(2/\epsilon_1), \Gamma((p-\delta)/p)p\left(1 - \frac{3\epsilon_1}{2r-4} - \eta\epsilon_1\right)\right\}.$$

Hence for $\tau = \exp(c_1(\eta,p)n)$, we use the above estimate of $\mathbf{P}_p(I_{t+1}^c \cap O_t)$ and the relation between O_t and I_t to have

$$\begin{aligned} \mathbf{P}_p\left(\inf_{t\leq\tau}\left|\xi_t^1\right| < (1-\epsilon_1)n\right) &= \mathbf{P}_p\left(\cup_{t=1}^{\lfloor\tau\rfloor} I_t^c\right) \\ &= \sum_{t=0}^{\lfloor\tau\rfloor-1} \mathbf{P}_p(I_{t+1}^c \cap O_t) \leq C_1\tau\ \exp(-2c_1(\eta,p)n) = C_1\exp(-c_1(\eta,p)n), \end{aligned}$$

and we get the desired result. ∎

5 First order phase transition at p_c

In this section, we use Theorem 1 and 2 to prove Theorem 3.

Proof of Theorem 3. First we estimate the probability $\mathbf{P}_p(\rho_n \geq \epsilon_2(p))$ for $p \in (p_c, 1)$. Let $\sigma_1 = \lceil \exp(n^{1/2}) \rceil$ and $\sigma_2(p) = \lceil C_0(p) \log n \rceil$, where $C_0(p)$ is as in Theorem 2. Depending on the fate of the process ξ_t^1 at time $\sigma_1 + \sigma_2$ and whether $G_n \in \mathcal{G}$ or not, where $\mathcal{G}$ is defined in Theorem 2, we have

$$\begin{aligned} & \mathbf{P}_p(\rho_n < \epsilon_2(p)) = \mathbf{P}_p(|\xi_{\sigma_1}^1| < \epsilon_2(p)n) \\ \leq \; & \mathbf{P}_p(\xi_{\sigma_1+\sigma_2}^1 = \emptyset) + \tilde{\mathbb{E}} P_{G_m,p}(|\xi_{\sigma_1}^1| < \epsilon_2(p)n \;,\; \xi_{\sigma_1+\sigma_2}^1 \neq \emptyset) \\ \leq \; & \mathbf{P}_p(\xi_{\sigma_1+\sigma_2}^1 = \emptyset) + \tilde{\mathbb{E}} \mathbf{1}_{\mathcal{G}^c} + \tilde{\mathbb{E}} \left[\mathbf{1}_{\mathcal{G}} P_{G_m,p}(|\xi_{\sigma_1}^1| < \epsilon_2(p)n \;,\; \xi_{\sigma_1+\sigma_2}^1 \neq \emptyset) \right]. \end{aligned} \tag{5.1}$$

By the definition of p_c in (1.3), the first term in the right side of (5.1) is $o(1)$ for $p \in (p_c, 1)$. By the estimate in (ii) of Theorem 2, the second term is also $o(1)$. To bound the third term in (5.1) we use Markov property of $P_{G_n,p}$ and the estimate in (i) of Theorem 2 to have

$$\begin{aligned} \mathbf{1}_{\mathcal{G}} P_{G_m,p}(|\xi_{\sigma_1}^1| < \epsilon_2(p) \;,\; \xi_{\sigma_1+\sigma_2}^1 \neq \emptyset) &= \sum_{A:|A|<\epsilon_2(p)n} P_{G_m,p}(\xi_{\sigma_1}^1 = A) \mathbf{1}_{\mathcal{G}} P_{G_n,p}(\xi_{\sigma_2}^A \neq \emptyset) \\ &\leq o(1) \sum_{A:|A|<\epsilon_2(p)n} P_{G_m,p}(\xi_{\sigma_1}^1 = A). \end{aligned}$$

Combining the last three observations,

$$\mathbf{P}_p(\rho_n < \epsilon_2(p)) \leq o(1) + o(1) + o(1) \sum_{A:|A|<n\epsilon_2(p)} \tilde{\mathbb{E}} P_{G_n,p}(\xi_{\sigma_1}^1 = A) = o(1).$$

Since $p_c < 1$ by Theorem 1 and $\epsilon_2(p) > 0$ for $p \in (0,1)$ and $\epsilon_2(\cdot)$ is a decreasing continuous function by Theorem 2, $\epsilon_2(p_c) > 0$ and for any $\delta \in (0, \epsilon_2(p_c))$, there exists $p' > p_c$ such that $\epsilon_2(p') > \epsilon_2(p_c) - \delta$.

Therefore, using the fact that $\epsilon_2(\cdot)$ is a decreasing function and the stochastic monotonicity of the probability distributions $\mathbf{P}_p, p \in [0,1]$, which is discussed in the introduction before Theorem 3, for any $p \in (p_c, 1]$

$$\begin{aligned} \mathbf{P}_p(\rho_n > \epsilon_2(p_c) - \delta) &\geq \mathbf{P}_p(\rho_n > \epsilon_2(p')) \\ &\geq \mathbf{P}_{p \wedge p'}(\rho_n \geq \epsilon_2(p \wedge p')) = 1 - o(1), \end{aligned}$$

where $p \wedge p' = \min\{p, p'\} > p_c$. So letting $n \to \infty$ the desired result follows. $\square$

6 Proof of the first isoperimetric inequality

In this section, we present the proof of the isoperimetric inequality in Proposition 1.

Proof of Proposition 1. In view of (2.2), it suffices to estimate the probability $\mathbb{P}[H(m,(r-1-\eta)m)$, where $H(m,k)=\{\exists\, U\subset V_n : |U|=m, |U^{*1}|\le k\}$. Recall the definitions of U_0 and U_1 from (2.1). We need some more notations to proceed. Given $\eta>0$ define the following events for a subset $U\subset V_n$.

$$\begin{aligned} &A_U := \{|U_1| \ge (\eta/2r)|U|\}, && B_U := \{|U^{*1}| \le (r-1-\eta)|U|\}, \\ &D_U := \{e(U,U^c) \le (r-2-\eta)|U|\}. \end{aligned} \tag{6.1}$$

There are three steps in the proof.

Step 1: Our first step is to estimate the probability that there is a subset U of vertices of size m for which $B_U\cap A_U^c$ occurs. On the event A_U^c, $|U_0|>(1-\eta/2r)|U|$ and so $e(U,U^c)\ge r|U_0|\ge(r-\eta/2)|U|$. Also on the event B_U, $|\partial U|\le|U^{*1}|\le(r-1-\eta)|U|$. From these two observations we have

$$\begin{aligned} \mathbb{P}(B_U\cap A_U^c) &\le \mathbb{P}(\{|\partial U|\le(r-1-\eta)|U|\}\cap\{e(U,U^c)\ge(r-\eta/2)|U|\}) \\ &\le \mathbb{P}(e(U,U^c)-|\partial U|\ge(1+\eta/2)|U|). \end{aligned} \tag{6.2}$$

Combining (6.2) with the bound in (ii) of Lemma 8.4,

$$\text{if } |U|=m\le\epsilon_5 n, \text{ then } \mathbb{P}(B_U\cap A_U^c)\le\exp\left[-(1+\eta/2)m\log(n/m)+\Delta_2 m\right]. \tag{6.3}$$

Suppose

$$F_1 := \cup_{\{U\subset V_n:|U|=m\}}\left(B_U\cap A_U^c\right).$$

Using (6.3) and the inequality in Lemma 8.1,

$$\begin{aligned} \text{if } m\le\epsilon_5 n, \text{ then } \mathbb{P}(F_1) &\le \binom{n}{m}\exp\left[-(1+\eta/2)m\log(n/m)+\Delta_2 m\right] \\ &\le \exp\left[-(\eta/2)m\log(n/m)+(1+\Delta_2)m\right]. \end{aligned} \tag{6.4}$$

If m is small enough, then the above estimate is exponentially small, and so with high probability there is no subset U of size m for which $B_U\cap A_U^c$ occurs.

Step 2: Our next step is to estimate the probability that there is a subset U of vertices for which A_U occurs and $e(U_1,U_1^c)\le(r-2-\eta)|U_1|$. If A_U occurs for some subset U of size m, then $|U_1|\in[\eta m/2r, m]$. So we consider all possible subsets having size in that range, and let

$$F_2 := \cup_{\{W:(\eta/2r)m\le|W|\le m\}} D_W.$$

Then using Lemma 8.2 with $\alpha=1-(2+\eta)/r$ and the inequality in Lemma 8.1,

$$\begin{aligned} \mathbb{P}(F_2) &= \mathbb{P}\left(\cup_{m'\in[\eta m/2r,m]}\cup_{\{W:|W|=m'\}}\{e(W,W^c)\le(r-2-\eta)m'\}\right) \\ &\le \sum_{m'\in[\eta m/2r,m]}\binom{n}{m'}C_5\exp\left[-\left(\frac{2+\eta}{2}\right)m'\log(n/m')+\Delta_1 m'\right] \\ &\le \sum_{m'\in[\eta m/2r,m]} C_5\exp\left(-(\eta/2)m'\log(n/m')+(1+\Delta_1)m'\right). \end{aligned} \tag{6.5}$$

Noting that the function $\phi(\eta) = \eta\log(1/\eta)$ is increasing on $(0, 1/e)$ (see (8.2)), if $m \le n/e$, then for $m' \in [\eta m/2r, m]$, $m'\log(n/m') \ge (\eta m/2r)\log(2rn/\eta m)$. Using this inequality and the fact that $(\eta/2r)\log(2r/\eta) > 0$, we can bound each summand in (6.5) by $C_5\exp(-(\eta/2)(\eta/2r)m\log(n/m) + (1+\Delta_1)m)$. As there are fewer than m terms in the sum over m' in (6.5), if we use the inequality $m \le e^m$ for $m \ge 0$, and

$$\text{if } m \le n/e, \text{ then } \mathbb{P}(F_2) \le C_5\exp\left(-(\eta/2)(\eta/2r)m\log(n/m) + (2+\Delta_1)m\right). \tag{6.6}$$

If m is small enough, then the right-hand side of (6.6) is exponentially small, and so with high probability there is no subset U of size m for which A_U occurs and $e(U_1, U_1^c) \le (r-2-\eta)|U_1|$.
Step 3: Our final step is to estimate the probability that there is a subset U of size m for which B_U occurs assuming F_1 and F_2 do not occur. Noting that U^{*1} is a disjoint union of U_1 and ∂U, and $|U| = |U_0| + |U_1|$, a little arithmetic gives

$$\begin{aligned}|U^{*1}| =& |U_1| + |\partial U| \\ =& (r-1-\eta)|U| + |\partial U| - (r-2-\eta)|U_1| - (r-1-\eta)|U_0|.\end{aligned}$$

Letting

$$\Delta(U) = |\partial U| - (r-2-\eta)|U_1| - (r-1-\eta)|U_0|, \tag{6.7}$$

we see that if B_U occurs, then $\Delta(U)$ has to be negative. Also if $|U| = m$, then by the definition of F_1, $B_U \cap F_1^c \subset B_U \cap A_U$, and on the event $A_U \cap F_2^c$, $|U_1| \ge (\eta/2r)|U|$ and so $e(U_1, U_1^c) > (r-2-\eta)|U_1|$. Combining these two observations,

$$\mathbb{P}(B_U \cap F_1^c \cap F_2^c) \le \mathbb{P}(B_U \cap A_U \cap F_2^c) \le \mathbb{P}(\{\Delta(U) \le 0\} \cap \{e(U_1, U_1^c) > (r-2-\eta)|U_1|\}). \tag{6.8}$$

Now by the definitions of U_0 and U_1,

$$\begin{aligned}&e(U_0, U_0^c) = e(U_0, U^c) = r|U_0| \text{ and } e(U_1, U_1^c) = e(U_1, U^c), \text{ so that} \\ &e(U, U^c) = e(U_0, U^c) + e(U_1, U^c) = r|U_0| + e(U_1, U_1^c),\end{aligned} \tag{6.9}$$

and a little algebra shows that $\{\Delta(U) \le 0\} = \{e(U, U^c) - |\partial U| \ge (1+\eta)|U_0| + e(U_1, U_1^c) - (r-2-\eta)|U_1|\}$. Also $e(U_1, U_1^c) < r|U_1|$. So

$$\begin{aligned}&\mathbb{P}(\{\Delta(U) \le 0\} \cap \{e(U_1, U_1^c) > (r-2-\eta)|U_1|\}) \qquad (6.10)\\ &= \sum_{\gamma \in (0, 2+\eta)} \mathbb{P}\left(\{e(U_1, U_1^c) = (r-2-\eta+\gamma)|U_1|\} \cap \{e(U, U^c) - |\partial U| \ge (1+\eta)|U_0| + \gamma|U_1|\}\right).\end{aligned}$$

Combining (6.8) and (6.10), and recalling that $|U_1| \in [\eta m/2r, m]$,

$$\begin{aligned}&\text{if we write } R = r-2-\eta, \\ &\text{and if } r(\gamma, k) := \mathbb{P}(e(U_1, U_1^c) = (R+\gamma)|U_1|, |U_1| = k) \text{ and} \\ &s(\gamma, k) := \mathbb{P}\left(e(U, U^c) - |\partial U| \ge (1+\eta)|U_0| + \gamma|U_1| \;\middle|\; e(U_1, U_1^c) = (R+\gamma)|U_1|, |U_1| = k\right), \\ &\text{then } \mathbb{P}(B_U \cap F_1^c \cap F_2^c) = \sum_{\gamma \in (0, 2+\eta)} \sum_{k \in [\eta m/2r, m]} r(\gamma, k)\, s(\gamma, k).\end{aligned} \tag{6.11}$$

In view of (6.9), if $L = (R+\gamma)k + r(m-k)$, then $\{e(U_1, U_1^c) = (R+\gamma)|U_1|\} \cap \{|U_1| = k\} = \{e(U, U^c) = L\} \cap \{|U_1| = k\}$. So

$$s(\gamma, k) = \mathbb{P}(e(U, U^c) - |\partial U| \geq \gamma k + (1+\eta)(m-k) \mid e(U, U^c) = L, |U_1| = k).$$

Since under the conditional distribution $\mathbb{P}(\cdot|e(U, U^c) = L)$ all the size-L subsets of half-edges corresponding to U^c are equally likely to be paired with those corresponding to U, the conditional distribution of $e(U, U^c) - |\partial U|$ given $e(U, U^c)$ and $|U_1|$ does not depend on $|U_1|$. So we can drop the event $\{|U_1| = k\}$ from the last display and use (i) of Lemma 8.4 with η replaced by $(\gamma k + (1+\eta)(m-k))/m$ to have

$$s(\gamma, k) \leq \exp\left(-\{\gamma k + (1+\eta)(m-k)\}\log(n/m) + \Delta_2 m\right), \text{ when } m \leq \epsilon_5 n. \tag{6.12}$$

In order to estimate $r(\gamma, k)$, we again use (6.9) and recall that $R = (r - 2 - \eta)$ to have

$$\begin{aligned} r(\gamma, k) &= \mathbb{P}(e(U_1, U_1^c) = (R+\gamma)k, |U_1| = k) \\ &= \mathbb{P}(e(U, U^c) = (R+\gamma)k + r(m-k), |U_1| = k) \\ &\leq \mathbb{P}(e(U, U^c) = rm - (2+\eta-\gamma)k), \end{aligned}$$

Using Lemma 8.2 with $\alpha = 1 - (2+\eta-\gamma)k/rm$,

$$r(\gamma, k) \leq C_5 \exp\left(-\frac{2+\eta-\gamma}{2} k \log(n/m) + \Delta_1 m\right). \tag{6.13}$$

Combining (6.11), (6.12) and (6.13), if $m \leq \epsilon_5 n$, then

$$\begin{aligned} &\mathbb{P}(B_U \cap F_1^c \cap F_2^c) \\ \leq &\sum_{\gamma \in (0, 2+\eta)} \sum_{k \in [\eta m/2r, m]} C_5 \exp\left[-\left\{\left(\frac{2+\eta+\gamma}{2}\right)k + (1+\eta)(m-k)\right\}\log(n/m) + (\Delta_1 + \Delta_2)m\right]. \end{aligned}$$

Noting that there are fewer than rm terms in the sum over γ and at most m terms in the sum over k, and using the inequality $m^2 \leq e^m$ for $m \geq 0$, the above is

$$\begin{aligned} &\leq C_5 r m^2 \exp\left[-(1+\eta/2)m\log(n/m) + (\Delta_1 + \Delta_2)m\right] \\ &\leq C_5 r \exp\left[-(1+\eta/2)m\log(n/m) + (1 + \Delta_1 + \Delta_2)m\right]. \end{aligned} \tag{6.14}$$

Recalling the definition of $H(m, (r-1-\eta)m)$ and considering whether the events $F_i, i = 1, 2$, occur or not,

$$\begin{aligned} \mathbb{P}(H(m, (r-1-\eta)m)) &= \mathbb{P}\left(\cup_{\{U:|U|=m\}} B_U\right) \\ &\leq \mathbb{P}(F_1) + \mathbb{P}(F_2) + \mathbb{P}\left(\cup_{\{U:|U|=m\}}(B_U \cap F_1^c \cap F_2^c)\right) \\ &\leq \mathbb{P}(F_1) + \mathbb{P}(F_2) + \sum_{\{U:|U|=m\}} \mathbb{P}(B_U \cap F_1^c \cap F_2^c). \end{aligned}$$

Combining (6.4), (6.6) and (6.14), and using the inequality in Lemma 8.1 to estimate the number of terms in the sum, if $m \leq \min\{1/e, \epsilon_5(\eta)\}n$, then

$$\begin{aligned}
& \mathbb{P}(H(m,(r-1-\eta)m)) \\
\leq\ & \mathbb{P}(F_1)+\mathbb{P}(F_2)+\binom{n}{m}C_5 r \exp\left[-(1+\eta/2)m\log(n/m)+(1+\Delta_1+\Delta_2)m\right] \\
\leq\ & \mathbb{P}(F_1)+\mathbb{P}(F_2)+C_5 r \exp\left[-(\eta/2)m\log(n/m)+(2+\Delta_1+\Delta_2)m\right] \\
\leq\ & C_3\exp[-(\eta^2/4r)m\log(n/m)+(2+\Delta_1+\Delta_2)m], \qquad (6.15)
\end{aligned}$$

where $C_3 = 3\max\{1, C_5 r\}$. To clean up the result to have the one given in Proposition 1, choose ϵ_3' such that

$$(\eta^2/4r)\log(1/\epsilon_3')/2 = 2+\Delta_1+\Delta_2, \quad \text{and} \quad \epsilon_3(\eta) := \min\{1/e, \epsilon_5(\eta), \epsilon_3'(\eta)\}, \qquad (6.16)$$

where ϵ_5 is defined in (8.8). So for any $m \leq \epsilon_3 n$, the estimate in (6.15) holds, and

$$(\eta^2/4r)\log(n/m)/2 \geq (\eta^2/4r)\log(1/\epsilon_3')/2 = 2+\Delta_1+\Delta_2,$$

which gives the desired estimate for the probability in (2.3), and thereby, in view of (2.2), provides the required bound for the probability in Proposition 1.

To finish the proof of Proposition 1 it remains to check that $\epsilon_3(\cdot)$ is increasing. By the definition of $\epsilon_5(\cdot)$ in (8.8) and the properties of $\beta(\cdot,\cdot)$ in Lemma 8.3, $\epsilon_5(\cdot)$ is increasing. Also by the definition of ϵ_3' in (6.16), $\log(1/\epsilon_3'(\cdot))$ is decreasing and hence $\epsilon_3'(\cdot)$ is increasing. Since minimum of increasing functions is still increasing, we conclude from (6.16) that $\epsilon_3(\cdot)$ is increasing. □

7 Proof of the second isoperimetric inequality

In this section, we present the proof of the isoperimetric inequality in Proposition 2.

Proof of Proposition 2. Recall the definitions of W^i and $\beta_i(W)$ from (2.6). We need some more notations to proceed. Given $\eta > 0$, let

$$Q_W := \left\{\beta_0(W)+\beta_1(W) > \frac{3}{2(r-2)}+\eta\right\}, \quad R_W := \left\{\beta_1(W) > \frac{1+2\eta}{r-2}\right\}.$$

We divide the argument into three steps.

Step 1: Our first step is to estimate the probability that there is a subset $W \subset V_n$ of size m for which R_W occurs. Since each $x \in \partial W$ has $e(\{x\}, W) \geq 1$ and each $x \in W^1$ has $e(\{x\}, W) \geq r-1$,

$$e(W, W^c) \geq (r-1)|W^1| + (|\partial W| - |W^1|) = (r-2)|W^1| + |\partial W|, \qquad (7.1)$$

and so $R_W \subset \{e(W,W^c) - |\partial W| \geq (1+2\eta)|W|\}$. Therefore, using (ii) of Lemma 8.4

$$\text{if } |W| = m \leq \epsilon_5 n, \text{ then } \mathbb{P}(R_W) \leq \exp[-(1+2\eta)m\log(n/m)+\Delta_2 m]. \qquad (7.2)$$

Now if

$$M_1 := \cup_{\{W: |W|=m\}} R_W,$$

then using (7.2) and the inequality in Lemma 8.1,

$$\begin{aligned} \text{if } m \le \epsilon_5 n, \text{ then } \mathbb{P}(M_1) &\le \binom{n}{m} \exp[-(1+2\eta)m\log(n/m) + \Delta_2 m] \\ &\le \exp[-2\eta m \log(n/m) + (1+\Delta_2)m]. \end{aligned} \tag{7.3}$$

If m is small enough, the above estimate is exponentially small, which implies that with high probability there is no subset W of size m for which R_W occurs.

Step 2: Our next step is to estimate the probability that there is a subset $W \subset V_n$ for which $Q_W \cap R_W^c$ occurs and $e(W^0, (W^0)^c) \le (r-2-(2r-4)\eta)|W^0|$. If $Q_W \cap R_W^c$ occurs for some subset W of size m, then a little algebra shows that for $r \ge 4$,

$$\beta_0(W) \ge \frac{3}{2(r-2)} + \eta - \frac{1+2\eta}{r-2} \ge \frac{1}{2(r-2)},$$

and so $|W^0| \in [m/(2r-4), m]$. For this reason we consider all possible subsets having size in that range and let

$$M_2 := \cup_{\{U: |U| \in [m/(2r-4), m]\}} \{e(U, U^c) \le (r-2-(2r-4)\eta)|U|\}.$$

Applying Lemma 8.2, using the inequality in Lemma 8.1, and then using an argument similar to the one leading to (6.6),

$$\begin{aligned} &\text{if } m \le n/e, \text{ then} \\ \mathbb{P}(M_2) &= \mathbb{P}\left(\cup_{m' \in [m/(2r-4), m]} \cup_{\{U: |U|=m'\}} \{e(U,U^c) \le (r-2-(2r-4)\eta)|U|\}\right) \\ &\le \sum_{m' \in [m/(2r-4), m]} \binom{n}{m'} C_5 \exp\left[-\frac{2+(2r-4)\eta}{2} m' \log(n/m') + \Delta_1 m'\right] \\ &\le C_5 \exp\left[-\frac{\eta}{2} m \log(n/m) + (2+\Delta_1)m\right]. \end{aligned} \tag{7.4}$$

If m is small enough, then the right hand side of (7.4) is exponentially small, and so with high probability there is no subset W of size m for which $Q_W \cap R_W^c$ occurs, and $e(W^0, (W^0)^c) \le (r-2-(2r-4)\eta)|W^0|$.

Step 3: Our final step is to estimate the probability that there is a subset $W \subset V_n$ for which Q_W occurs assuming M_1 and M_2 do not occur. If $|W| = m$, then by the definition of M_1, $Q_W \cap M_1^c \subset Q_W \cap R_W^c$. On the event $Q_W \cap R_W^c \cap M_2^c$, $|W^0| \in [m/(2r-4), m]$ and so $e(W^0, (W^0)^c) > (r-2-(2r-4)\eta)|W^0|$. Also by the definition of W^0, $e(W^0, W^c) \le |W^0|$. Combining these three observations with the fact that $W^0 \subset W$, on the event $Q_W \cap M_1^c \cap M_2^c$,

$$\begin{aligned} e(W^0, W \setminus W^0) &= e(W^0, (W^0)^c) - e(W^0, W^c) \\ &\ge (r-2-(2r-4)\eta)|W^0| - |W^0| \\ &= (r-3-(2r-4)\eta)\beta_0(W)|W|. \end{aligned} \tag{7.5}$$

Next we see that W is a disjoint union of W^0 and $W \setminus W^0$, and $(W \setminus W^0)^c$ is a disjoint union of W^0 and W^c. So

$$\begin{aligned} e(W, W^c) &= e(W \setminus W^0, W^c) + e(W^0, W^c) \\ &= e(W \setminus W^0, (W \setminus W^0)^c) - e(W \setminus W^0, W^0) + e(W^0, W^c). \end{aligned} \tag{7.6}$$

Combining the inequalities in (7.5) and (7.6), recalling that $e(W \setminus W^0, (W \setminus W^0)^c) \leq r|W \setminus W^0|$, and again using the inequality $e(W^0, W^c) \leq |W^0|$, we see that on the event $Q_W \cap M_1^c \cap M_2^c$,

$$\begin{aligned} e(W, W^c) &\leq r|W \setminus W^0| - e(W^0, W \setminus W^0) + e(W^0, W^c) \\ &\leq [r - (2r-4)(1-\eta)\beta_0(W)]|W|. \end{aligned}$$

Therefore by (7.1),

$$\begin{aligned} |\partial W| &\leq e(W, W^c) - (r-2)|W^1| \\ &\leq [r - (2r-4)(1-\eta)\beta_0(W) - (r-2)\beta_1(W)]|W|. \end{aligned} \tag{7.7}$$

Now we show that $(2r-4)(1-\eta)\beta_0(W) + (r-2)\beta_1(W) > 2+\eta$ on the event $Q_W \cap M_1^c \cap M_2^c$. By the definition of M_1, $\beta_1(W) \leq (1+2\eta)/(r-2)$ on the event $Q_W \cap M_1^c \cap M_2^c$. So if $(2r-4)(1-\eta)\beta_0(W) + (r-2)\beta_1(W) \leq 2+\eta$ on the same event, then noting that the maximum value of $\beta_0 + \beta_1$ under the constraints (i) $(2r-4)(1-\eta)\beta_0 + (r-2)\beta_1 \leq 2+\eta$ and (ii) $\beta_1 \leq (1+2\eta)/(r-2)$ is attained when both constraints hold with equality, a little algebra shows that

$$\beta_1 + \beta_0 \leq \frac{1+2\eta}{r-2} + \frac{1}{2(r-2)} = \frac{3}{2(r-2)} + \frac{2}{r-2}\eta \leq \frac{3}{2(r-2)} + \eta$$

on the event $Q_W \cap M_1^c \cap M_2^c$. But the definition of Q_W contradicts that. So, on the event $Q_W \cap M_1^c \cap M_2^c$, we must have $(2r-4)(1-\eta)\beta_0(W) + (r-2)\beta_1(W) > 2+\eta$ and hence $|\partial W| < (r-2-\eta)|W|$ by (7.7). Thus $\mathbb{P}(Q_W \cap M_1^c \cap M_2^c) \leq \mathbb{P}(|\partial W| < (r-2-\eta)|W|)$. In order to estimate the right-hand side of the last inequality we apply Lemma 8.5 to have

$$\text{if } |W| = m \leq \epsilon_5 n, \text{ then } \mathbb{P}(Q_W \cap M_1^c \cap M_2^c) \leq C_7 \exp(-(1+\eta/4)m \log(n/m) + (1+\Delta_1+\Delta_2)m). \tag{7.8}$$

Recalling the definition of $F(m,k)$ and considering whether the events $M_i, i = 1, 2$, occur or not,

$$\begin{aligned} \mathbb{P}[F(m, [3/(2r-4)+\eta]m)] &= \mathbb{P}\left(\cup_{\{W:|W|=m\}} Q_W\right) \\ &\leq \mathbb{P}(M_1) + \mathbb{P}(M_2) + \mathbb{P}\left(\cup_{\{W:|W|=m\}} (Q_W \cap M_1^c \cap M_2^c)\right) \\ &\leq \mathbb{P}(M_1) + \mathbb{P}(M_2) + \sum_{\{W:|W|=m\}} \mathbb{P}(Q_W \cap M_1^c \cap M_2^c). \end{aligned}$$

Combining the probability bounds in (7.3), (7.4) and (7.8), using the inequality in Lemma 8.1 to estimate the number of terms in the sum, if $m \le \min\{1/e, \epsilon_5(\eta)\}n$, then

$$\begin{aligned}
& \mathbb{P}[F(m, [3/(2r-4)+\eta]m)] \\
\le\ & \mathbb{P}(M_1) + \mathbb{P}(M_2) + \binom{n}{m} C_7 \exp\left[-(1+\eta/4)m\log(n/m) + (1+\Delta_1+\Delta_2)m\right] \\
\le\ & \mathbb{P}(M_1) + \mathbb{P}(M_2) + C_7 \exp\left[-(\eta/4)m\log(n/m) + (2+\Delta_1+\Delta_2)m\right] \\
\le\ & C_4 \exp[-(\eta/4)m\log(n/m) + (2+\Delta_1+\Delta_2)m], \qquad (7.9)
\end{aligned}$$

where $C_4 = 3\max\{1, C_7\}$. To clean up the result to have the one given in Proposition 2, choose $\epsilon_4'(\eta)$ such that

$$(\eta/8)\log(1/\epsilon_4') = (2+\Delta_1+\Delta_2), \quad \text{and} \quad \epsilon_4 := \min\{1/e, \epsilon_5(\eta), \epsilon_4'(\eta)\}, \qquad (7.10)$$

where ϵ_5 is defined in (8.8). So for any $m \le \epsilon_4 n$, the estimate in (7.9) holds, and

$$(\eta/8)\log(n/m) \ge (\eta/8)\log(1/\epsilon_4') = (2+\Delta_1+\Delta_2),$$

which gives the desired result. □

8 Probability estimates for $e(U, U^c)$ and $|\partial U|$

We begin with a simple estimate for the number of subsets of V_n of size m.

Lemma 8.1. *The number of subsets of V_n of size m is at most* $\exp(m\log(n/m) + m)$.

Proof. The number of subsets of V_n of size m is $\binom{n}{m}$. Using the inequalities $n(n-1)\cdots(n-m+1) \le n^m$ and $e^m > m^m/m!$,

$$\binom{n}{m} \le \frac{n^m}{m!} \le \left(\frac{ne}{m}\right)^m = \exp(m\log(n/m) + m).$$

□

In order to study the distribution of $|\partial U|$, the first step is to estimate $e(U, U^c)$. Because of the symmetries of our random graph G_n, the distribution of $e(U, U^c)$ under $\mathbb{P}$ depends on U only through $|U|$.

Lemma 8.2. *Let U be any subset of V_n with $|U| = m$. Then for any $\alpha \in (0,1)$,*

$$\mathbb{P}(e(U, U^c) \le \alpha r|U|) \le C_5 \exp\left(-\frac{r}{2}(1-\alpha)m\log(n/m) + \Delta_1 m\right)$$

for some constants C_5 and Δ_1.

Proof. Let $f(u)$ be the number of ways of pairing u objects. Then

$$f(u) = \frac{u!}{(u/2)!2^{u/2}}.$$

If $p(m,s) = \mathbb{P}(e(U,U^c) = s)$, then we have

$$p(m,s) \leq \binom{rm}{s}\binom{r(n-m)}{s} s! \frac{f(rm-s)f(r(n-m)-s)}{f(rn)}.$$

To see this, recall that we construct the graph G_n by pairing the half-edges at random, which can be done in $f(rn)$ many ways as there are rn half-edges. We can choose the left endpoints of the edges from U in $\binom{rm}{s}$ many ways, the right endpoints from U^c in $\binom{r(n-m)}{s}$ many ways, and pair them in $s!$ many ways. The remaining $(rm-s)$ many half-edges of U can be paired among themselves in $f(rm-s)$ many ways. Similarly the remaining $(r(n-m)-s)$ many half-edges of U^c can be paired among themselves in $f(r(n-m)-s)$ many ways.

Write $D = rn, k = rm$ and $s = \eta k$ for $\eta \in [0,1]$. Combining the bounds of (6.3.4) and (6.3.5) of [Dur07] we get

$$p(m,s) \leq C_6 k^{1/2} \left(\frac{e^2}{\eta}\right)^{\eta k} \left(\frac{k}{D}\right)^{k(1-\eta)/2} \left(1 - \frac{(1+\eta)k}{D}\right)^{(D-(1+\eta)k)/2} \tag{8.1}$$

for some constant C_6. Now

$$\text{if } \phi(\eta) = \eta \log(1/\eta), \text{ then } \phi'(\eta) = -(1+\log\eta) \text{ and } \phi''(\eta) = -\frac{1}{\eta}. \tag{8.2}$$

So $\phi(\cdot)$ is a concave function and its derivative vanishes at $1/e$. This shows that the function $\phi(\cdot)$ is maximized at $1/e$, and hence $(1/\eta)^\eta = e^{\phi(\eta)} \leq e^{1/e}$ for $\eta \in [0,1]$. So $(e^2/\eta)^{\eta k} \leq C^k$ for $C = \exp(2 + 1/e)$. If we ignore the last term of (8.1), which is ≤ 1, then we have

$$\begin{aligned}\mathbb{P}(e(U,U^c) \leq \alpha rm) = \sum_{s=1}^{\lfloor \alpha rm \rfloor} p(m,s) \leq & \sum_{\{\eta:\, \eta rm \in \mathbb{N},\, \eta \leq \alpha\}} C_6 (rm)^{1/2} C^{rm} \left(\frac{m}{n}\right)^{rm(1-\eta)/2} \\ \leq & C_6 r^{3/2} m^{3/2} C^{rm} \left(\frac{m}{n}\right)^{r(1-\alpha)m/2},\end{aligned}$$

as there are at most rm terms in the sum and $(m/n)^{1-\eta} \leq (m/n)^{1-\alpha}$ for $\eta \leq \alpha$. The above bound is

$$\leq C_5 \exp\left(-\frac{r}{2}(1-\alpha)m\log(n/m) + rm\log C + 3m/2\right),$$

and we get the desired result with $C_5 = C_6 r^{3/2}$ and $\Delta_1 = r\log C + 3/2$. $\square$

Lemma 8.2 gives an upper bound for the probability that $e(U,U^c)$ is small. Our next goal is to estimate the difference between $e(U,U^c)$ and $|\partial U|$. In order to do that, first we need the following large deviation probability estimate.

Lemma 8.3. *If $T_1, T_2, \ldots$ are independent random variables and $T_i \sim$ Geometric(p_i) with $p_i = (n-i+1)/n$, then for any $u > 0$ and $\eta \in (0, u)$ there are positive constants Δ_2 and $\beta = \beta(u, \eta)$ such that for large enough n and any $m < \beta n$,*

$$P\left(T_1 + T_2 + \cdots + T_{\lfloor (u-\eta)m \rfloor} > um\right) \leq \exp\left[-\eta m \log(n/m) + \Delta_2 m\right].$$

Moreover, $\beta(u,\eta) \downarrow 0$ as $\eta \downarrow 0$ and for fixed η, $\beta(u,\eta)$ is a decreasing function of u.

Proof. Let $q_i = 1 - p_i = (i-1)/n$. Then for $\theta < \log(1/q_i)$,

$$E\left[e^{\theta T_i}\right] = \sum_{k=1}^{\infty} p_i q_i^{k-1} e^{\theta k} = \frac{p_i e^{\theta}}{1 - q_i e^{\theta}}.$$

Let $\epsilon = m/n$, $\theta > 0$ and $\epsilon e^{\theta} < 1/(u-\eta)$ so that $Ee^{\theta T_i}$ is finite for $i = 1, 2, \ldots, \lfloor (u-\eta)m \rfloor$. Using Markov inequality

$$P\left(T_1 + \cdots + T_{\lfloor (u-\eta)m \rfloor} > um\right) \leq \exp\left[-\theta u m\right] \prod_{i=1}^{\lfloor (u-\eta)m \rfloor} Ee^{\theta T_i}.$$

Using $\epsilon = m/n$ and the formula for $E\exp(\theta T_i)$, a little arithmetic shows that the above is

$$\begin{aligned}
&\leq \exp\left[-\theta u \epsilon n + \sum_{i=1}^{\lfloor (u-\eta)\epsilon n \rfloor} \log \frac{p_i e^{\theta}}{1 - q_i e^{\theta}}\right] \\
&\leq \exp\left[-\theta \eta \epsilon n + n \cdot \frac{1}{n} \sum_{i=1}^{\lfloor (u-\eta)\epsilon n \rfloor} \log \frac{1 - (i-1)/n}{1 - (i-1)e^{\theta}/n}\right]. \qquad (8.3)
\end{aligned}$$

Since $e^{\theta} > 1$, it can be verified that the function $g(x) = \log[(1-x)/(1-xe^{\theta})]$ is increasing, so that we can bound the Riemann sum for the function $g(x)$ in (8.3) by the corresponding integral. Thus the above is

$$\leq \exp\left[-\theta\eta\epsilon n + n\left(\int_0^{(u-\eta)\epsilon} \log(1-x)dx - \int_0^{(u-\eta)\epsilon} \log(1 - xe^{\theta})dx\right)\right]. \qquad (8.4)$$

To bound the last quantity we let

$$h(\theta, u, \eta, \epsilon) = \theta\eta\epsilon - \left(\int_0^{(u-\eta)\epsilon} \log(1-x)dx - \int_0^{(u-\eta)\epsilon} \log(1 - xe^{\theta})dx\right).$$

Clearly $h(0, u, \eta, \epsilon) = 0$. We want to maximize h with respect to θ keeping all the other parameters fixed. Changing the variables $y = 1 - x$ and $z = 1 - xe^{\theta}$,

$$\begin{aligned}
h &= \theta\eta\epsilon - \left(\int_{1-(u-\eta)\epsilon}^{1} \log y \, dy - e^{-\theta} \int_{1-(u-\eta)\epsilon e^{\theta}}^{1} \log z \, dz\right) \\
&= \theta\eta\epsilon - \big(-(1 - (u-\eta)\epsilon)\log(1 - (u-\eta)\epsilon) \\
&\qquad + e^{-\theta}\left(1 - (u-\eta)\epsilon e^{\theta}\right)\log\left(1 - (u-\eta)\epsilon e^{\theta}\right)\big), \qquad (8.5)
\end{aligned}$$

where to evaluate the integrals we recall $(x\log x - x)' = \log x$.

$$\begin{aligned}\partial h/\partial\theta &= \eta\epsilon + e^{-\theta}\left(1-(u-\eta)\epsilon e^{\theta}\right)\log\left(1-(u-\eta)\epsilon e^{\theta}\right)\\ &\qquad + e^{-\theta}(u-\eta)\epsilon e^{\theta}\log\left(1-(u-\eta)\epsilon e^{\theta}\right) - e^{-\theta}\left(1-(u-\eta)\epsilon e^{\theta}\right)\frac{-(u-\eta)\epsilon e^{\theta}}{1-(u-\eta)\epsilon e^{\theta}}\\ &= \eta\epsilon + (u-\eta)\epsilon + e^{-\theta}\log\left(1-(u-\eta)\epsilon e^{\theta}\right) = u\epsilon + e^{-\theta}\log\left(1-(u-\eta)\epsilon e^{\theta}\right).\end{aligned}$$

$\partial h/\partial\theta = 0$ implies $\exp(-u\epsilon e^{\theta}) = 1-(u-\eta)\epsilon e^{\theta}$. Letting

$$\beta = \beta(u,\eta) \text{ be the unique positive number satisfying } e^{-u\beta} = 1-(u-\eta)\beta, \tag{8.6}$$

$\partial h/\partial\theta > 0$ if $\epsilon e^{\theta} < \beta$. (8.6) suggests that $\beta \in (0, 1/(u-\eta))$. So for fixed u, η, $\epsilon < \beta(u,\eta)$ and $\theta^* := \log(\beta/\epsilon)$, $\theta^* > 0$ with $\epsilon e^{\theta^*} < 1/(u-\eta)$, and the function h is maximized at θ^*. Plugging the value of θ^* in (8.5),

$$h = \eta\epsilon\log(\beta/\epsilon) + (1-(u-\eta)\epsilon)\log(1-(u-\eta)\epsilon) - \frac{\epsilon}{\beta}(1-(u-\eta)\beta)\log(1-(u-\eta)\beta).$$

Noting that the function

$$\psi(\delta) := \frac{(1-\delta)\log(1-\delta)}{\delta} \text{ satisfies } \psi'(\delta) = \frac{-\delta-\log(1-\delta)}{\delta} > 0, \text{ and } \psi(\delta) \to \begin{cases} -1 \text{ if } \delta\to 0 \\ 0 \text{ if } \delta \to 1\end{cases},$$

$[\psi(\delta) - \psi(\delta')] \geq -1$ for $\delta, \delta' \in (0,1)$, and so

$$\begin{aligned} h &= \eta\epsilon\log(1/\epsilon) + \eta\epsilon\log\beta + (u-\eta)\epsilon[\psi((u-\eta)\epsilon) - \psi((u-\eta)\beta)]\\ &\geq \eta\epsilon\log(1/\epsilon) - c_2(u,\eta)\epsilon, \end{aligned} \tag{8.7}$$

where $c_2(u,\eta) = u - \eta + \eta\log(1/\beta(u,\eta))$.

To see that $\beta(u,\eta)$ has the desired properties, note that if $\varphi_x(u,\eta) := e^{-ux} - 1 + (u-\eta)x$, then for $x > 0$, $\partial\varphi_x/\partial u = -xe^{-ux} + x > 0$ and $\partial\varphi_x/\partial\eta = -x < 0$. If we put $x = \beta(u,\eta)$, use (8.6), and note that $\varphi_x(u,\eta) \leq 0$ if and only if $0 \leq x \leq \beta(u,\eta)$, then

$$\begin{aligned} &\text{for } u' > u, \varphi_{\beta(u,\eta)}(u',\eta) > \varphi_{\beta(u,\eta)}(u,\eta) = 0, \text{ and so we must have } \beta(u',\eta) < \beta(u,\eta),\\ &\text{for } \eta' < \eta, \varphi_{\beta(u,\eta)}(u,\eta') > \varphi_{\beta(u,\eta)}(u,\eta) = 0, \text{ and so we must have } \beta(u,\eta') < \beta(u,\eta). \end{aligned}$$

To ensure that $\beta(u,\eta) \downarrow 0$ as $\eta \downarrow 0$, see that if $\beta(u,0) := \lim_{\eta\to 0}\beta(u,\eta)$, then using continuity of $\beta(u,\eta)$ and (8.6), $\exp(-u\beta(u,0)) = 1 - u\beta(u,0)$ and so $\beta(u,0) = 0$.

Using the properties of $\beta(u,\eta)$ we can show that $c_2(u,\beta)$ is bounded above as η and u vary. From the inequality $e^{-y} \geq 1-y$ we have $1 - e^{-x} = \int_0^x e^{-y}\,dy \geq \int_0^x (1-y)\,dy = x - x^2/2$ for any $x \geq 0$. In view of (8.6), using the last inequality we see that

$$1-(u-\eta)\beta = e^{-u\beta} \leq 1 - u\beta + \frac{u^2\beta^2}{2}, \text{ which implies } \beta \geq \frac{2\eta}{u^2} \text{ and so } c_2(u,\eta) \leq u - \eta + \eta\log\left(\frac{u^2}{2\eta}\right),$$

and $\limsup_{\eta\to 0} c_2(u,\eta) \le u$. In the other direction, $\beta(u,\eta) \to \infty$ as $\eta \to u$, since for any $\beta_0 > 0$ we can choose $\eta_0 \in (0,u)$ so that $1-(u-\eta_0)\beta_0 > e^{-u\beta_0}$ (e.g. choose η_0 satisfying $1-(u-\eta_0)\beta_0 = (1+e^{-u\beta_0})/2$) to make sure $\beta(u,\eta_0) > \beta_0$. Thus $c_2(u,\eta) \to -\infty$ as $\eta \to u$. From the behavior of $c_2(u,\eta)$ when η is close to 0 and u , and noting that $c_2(u,\eta)$ depends continuously on η,

$$c_0(u) := \max\{c_2(u,\eta) : \eta \in (o,u)\} < \infty.$$

Next we recall that $e(U,U^c) \le r|U|$ so that $u \in [0,r]$. Since $\beta(u,\eta)$ is decreasing in u, recalling the definitions of $c_2(u,\eta)$ and $c_0(u)$ it is easy to see that for fixed η, $c_2(u,\eta)$ is increasing in u, and hence so is $c_0(u)$. Therefore,

$$\text{if } \Delta_2 := c_0(r), \text{ then } c_2(u,\eta) \le c_0(u) \le \Delta_2 \text{ for any } 0 < \eta < u \le r.$$

Coming back to estimate h, we can convert (8.7) to

$$h \ge \eta\epsilon\log(1/\epsilon) - \Delta_2\epsilon.$$

Plugging the bound on h and $\epsilon = m/n$ in (8.4) we get

$$P\left(T_1 + \cdots + T_{\lfloor (u-\eta)m\rfloor} > um\right) \le \exp(-\eta m \log(n/m) + \Delta_2 m).$$

which completes the proof of Lemma 8.3 □

Now we use Lemma 8.3 to get an upper bound for the probability that the difference between $e(U,U^c)$ and $|\partial U|$ is large.

Lemma 8.4. *If U is a subset of vertices of G_n such that $|U| = m$, then for any $\eta > 0$, $u \in (\eta, r]$ and Δ_2 as in Lemma 8.3, there is a constant $\epsilon_5 = \epsilon_5(\eta) > 0$ such that for large enough n and $m < \epsilon_5 n$,*

$$\begin{array}{lll} (i) & \mathbb{P}\left(|\partial U| \le (u-\eta)|U| \mid e(U,U^c) = u|U|\right) & \le \exp(-\eta m\log(n/m) + \Delta_2 m), \\ (ii) & \mathbb{P}\left(e(U,U^c) - |\partial U| > \eta|U|\right) & \le \exp(-\eta m\log(n/m) + \Delta_2 m). \end{array}$$

Proof. Since $|U^c| = n-m$, there are $r(n-m)$ many half-edges corresponding to U^c. In order to have $e(U,U^c) = um$, we need to choose um half-edges corresponding to U^c and pair them with the same number of half-edges corresponding to U. Since the half-edges are paired randomly under the probability distribution $\mathbb{P}$, all the subsets of half-edges corresponding to U^c of size um are equally likely to be chosen under the conditional probability distribution $\mathbb{P}(\cdot|e(U,U^c) = um)$. Noting that the subset of size um, which is obtained by choosing um objects one at a time from a set of size $r(n-m)$ uniformly at random without replacement, has uniform distribution over all possible subsets of that size, we can assume that the half-edges corresponding to U^c mentioned above are chosen one by one uniformly at random without replacement.

Suppose R_i half-edges are chosen by the time i many distinct vertices are chosen. Let $T_1' = R_1 = 1$ and $T_i' = R_i - R_{i-1}$ for $i \ge 2$. Since each vertex has r half-edges, $R_{i+1} \le 1 + ri$ and $e(U,U^c) \le r|U|$ so that $u \le r$. A little arithmetic gives that for large enough n,

$$\frac{n}{r^2+r+1} \le \frac{n-1}{r^2+r} \le \frac{n-1}{ru+r} \text{ so that for } m \le \frac{n}{r^2+r+1} \text{ and } i = 1,\ldots,um, ri+1+rm \le n.$$

Combining these inequalities, after choosing the i^{th} distinct vertex the failure probability to choose the $(i+1)^{th}$ distinct vertex at any step is

$$\leq \frac{ri-i}{r(n-m)-ri-1} \leq \frac{i}{n} \text{ for } i \leq um.$$

Then, on the event $\{e(U,U^c)=um\}$, the T_i' can be coupled with geometric random variables T_i with failure probability $(i-1)/n$ so that $T_i' \leq T_i$. So

$$\begin{aligned} \mathbb{P}(R_{\lfloor (u-\eta)m \rfloor} > um | e(U,U^c)=um) &= \mathbb{P}\left(T_1' + \cdots + T'_{\lfloor (u-\eta)m \rfloor} > um | e(U,U^c)=um\right) \\ &\leq P\left(T_1 + \cdots + T_{(u-\eta)m} > um\right), \end{aligned}$$

when $m \leq n/(1+r+r^2)$. If we let

$$\epsilon_5(\eta) = \min\{1/(1+r+r^2), \beta(r,\eta/2)\}, \tag{8.8}$$

where β is defined in Lemma 8.3, then for $m \leq \epsilon_5 n$ we have the above inequality and can use the probability estimate of Lemma 8.3 as $\beta(u,\eta) > \beta(r,\eta)$. From those two inequalities we conclude that

$$\begin{aligned} \mathbb{P}\left(|\partial U| < (u-\eta)m \mid e(U,U^c)=um\right) &\leq \mathbb{P}(R_{\lfloor (u-\eta)m \rfloor} > um | e(U,U^c)=um) \\ &\leq \exp(-\eta m \log(n/m) + \Delta_2 m) \end{aligned}$$

for $m \leq \epsilon_5 n$, which completes the proof of (i).

To prove (ii), recall that $e(U,U^c) \leq rm$. So based on $e(U,U^c)$ we have

$$\begin{aligned} & \mathbb{P}(e(U,U^c) - |\partial U| \geq \eta m) \\ \leq & \sum_{u \in (\eta,r]:\, um \in \mathbb{N}} \mathbb{P}(e(U,U^c) - |\partial U| \geq \eta m, e(U,U^c)=um) \\ = & \sum_{u \in (\eta,r]:\, um \in \mathbb{N}} \mathbb{P}(e(U,U^c) - |\partial U| \geq \eta m | e(U,U^c)=um) \mathbb{P}(e(U,U^c)=um). \end{aligned} \tag{8.9}$$

If $m \leq \epsilon_5 n$, we can use (i) to bound the first terms of the summands in the right-hand side of (8.9) and have

$$\begin{aligned} & \mathbb{P}(e(U,U^c) - |\partial U| \geq \eta m) \\ \leq & \exp(-\eta m \log(n/m) + \Delta_2 m) \sum_{u \in (\eta,r]:\, um \in \mathbb{N}} \mathbb{P}(e(U,U^c)=um) \\ \leq & \exp(-\eta m \log(n/m) + \Delta_2 m). \end{aligned}$$

□

Lemma 8.4 gives an upper bound for the probability that the difference between $|\partial U|$ and $e(U,U^c)$ is large. Now we use Lemma 8.2 and 8.4 to estimate the probability that $|\partial U|$ is smaller than $(r-2)|U|$.

Lemma 8.5. *Let $U \subset V_n$ be such that $|U| = m$ and $\eta > 0$. For the constants Δ_1 of Lemma 8.2, ϵ_5 and Δ_2 of Lemma 8.4, if n is large enough and $m \le \epsilon_5 n$, then*

$$\mathbb{P}(|\partial U| \le (r-2-\eta)|U|) \le C_7 \exp[-(1+\eta/4)m\log(n/m) + (1+\Delta_1+\Delta_2)m]$$

for some constant C_7.

Proof. First we estimate the probability $\mathbb{P}(|\partial U| = (r-2-\eta')|U|)$ when $\eta' \ge \eta$. Noting that $|\partial U| \le e(U,U^c) \le r|U|$ for any $U \subset V_n$,

$$\begin{aligned}&\mathbb{P}(|\partial U| = (r-2-\eta')|U|)\\ =&\sum_{\gamma\in[0,2+\eta']} \mathbb{P}(|\partial U| = (r-2-\eta')|U|, e(U,U^c) = (r-2-\eta'+\gamma)|U|).\end{aligned} \tag{8.10}$$

For the summands with $\gamma \ge \eta'/2$, we write each summand as the product of two terms

$$\mathbb{P}(e(U,U^c) = (r-2-\eta'+\gamma)|U|)\,\mathbb{P}(|\partial U| = (r-2-\eta')|U|| e(U,U^c) = (r-2-\eta'+\gamma)|U|).$$

We can use Lemmas 8.2 to estimate the first term above. For the second term, note that by the definition of ϵ_5 in (8.8) and the properties of $\beta(\cdot,\cdot)$ in Lemma 8.3, if $\gamma \ge \eta'/2$, then $\beta(r-2-\eta'+\gamma,\gamma) \ge \beta(r,\eta/2) \ge \epsilon_5$. So if $|U| = m \le \epsilon_5 n$, we can use (i) of Lemma 8.4 to estimate the second term in the last display, and have

$$\begin{aligned}&\mathbb{P}(|\partial U| = (r-2-\eta')|U|, e(U,U^c) = (r-2-\eta'+\gamma)|U|)\\ \le& C_5 \exp\left[-\left(\frac{2+\eta'-\gamma}{2}\right) m\log(n/m) + \Delta_1 m\right] \cdot \exp(-\gamma m \log(n/m) + \Delta_2 m).\end{aligned}$$

As there are fewer than rm terms in the sum over γ and each term has the same upper bound $C_5 \exp(-(1+\eta'/2)m\ log(n/m) + (\Delta_1+\Delta_2)m)$, noting that $m \le e^{m/2}$ for $m \ge 0$,

$$\begin{aligned}&\sum_{\gamma\in[\eta'/2,2+\eta']} \mathbb{P}(|\partial U| = (r-2-\eta')|U|, e(U,U^c) = (r-2-\eta'+\gamma)|U|)\\ \le&\ rC_5 \exp\left[-\left(\frac{2+\eta}{2}\right) m\log(n/m) + (1/2+\Delta_1+\Delta_2)m\right].\end{aligned} \tag{8.11}$$

For the summands in (8.10) with $\gamma < \eta'/2$, we can ignore one of the two events and use Lemma 8.2 to have

$$\begin{aligned}&\sum_{\gamma\in[0,\eta'/2)} \mathbb{P}(|\partial U| = (r-2-\eta')|U|, e(U,U^c) = (r-2-\eta'+\gamma)|U|) \qquad (8.12)\\ \le&\ \mathbb{P}(e(U,U^c) \le (r-2-\eta'/2)|U|) \le C_5 \exp\left(-\frac{2+\eta'/2}{2} m\log(n/m) + \Delta_1 m\right).\end{aligned}$$

Combining (8.11) and (8.13), noting that there are at most rm terms in the sum over η' below, and again using the inequality $m \le e^{m/2}$ for $m \ge 0$,

$$\begin{aligned}
&\mathbb{P}(|\partial U| \le (r-2-\eta)|U|) = \sum_{\eta' \in [\eta, r-2]} \mathbb{P}(|\partial U| = (r-2-\eta')|U|) \\
&\le \sum_{\eta' \in [\eta, r-2]} (C_5 + C_5 r) \exp(-(1+\eta'/4) m \log(n/m) + (1/2 + \Delta_1 + \Delta_2) m) \\
&\le r C_5 (1+r) \exp(-(1+\eta/4) m \log(n/m) + (1 + \Delta_1 + \Delta_2) m),
\end{aligned}$$

and we get the desired result with $C_7 = C_5 r(1+r)$. □

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References

[CD] S. Chatterjee and R. Durrett, *Persistence of Activity in Threshold Contact Processes, an "Annealed Approximation" of Random Boolean Networks*, Submitted, http://arxiv.org/abs/0911.5339.

[DJ07] R. Durrett and P. Jung, *Two phase transitions for the contact process on small worlds*, Stochastic Processes and their Applications **117** (2007), no. 12, 1910–1927.

[DN94] R. Durrett and C. Neuhauser, *Particle systems and reaction-diffusion equations*, The Annals of Probability **22** (1994), no. 1, 289–333.

[Dur07] R. Durrett, *Random graph dynamics*, 2007.

[JŁR00] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Citeseer, 2000.

[Lig85] T.M. Liggett, *Interacting Particle Systems. 1985*, 1985.

[Lig99] ———, *Stochastic interacting systems: contact, voter, and exclusion processes*, Springer Verlag, 1999.